Distance Geometry III - Laman's Theorem and the Pebble Game.

We ended last time with excited babbling about the following question: given some entries of a distance matrix, when can you choose values for the missing entries which make the completed matrix Euclidean?

In 2d, we can say quite a bit. We start by counting.

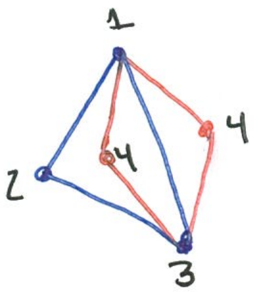
n points in $\mathbb{R}^2$ have $2n$ degrees of freedom.
every ~~framework~~ point cloud has 3 independent rigid motions which don't change any lengths

$\Rightarrow$ we guess a ~~framework~~ distance matrix must have at least $2n-3$ ~~d~~ distances specified to ~~det~~ determine the remaining d_{ij}.

In fact, these distances must be spread out "evenly" in order to decide the remaining distances.

We now go back to rigidity theory to characterize those ~~choices~~ subsets of distances which, if fixed, allow only isolated realizations.

Example. 4 points, fixing all distances between 1, 2, and 3 and the distances ~~bet~~ d_{14} and d_{34} yields a par distance matrix with 2 completions.



Definition. A subset of d_{ij} determines a graph on the vertices p_i. We say the graph is <u>generically rigid</u> if the set of Euclidean completions of generic (realizable) choices for these distances consists of isolated points.

~~Laman's Theorem. The g~~

~~We say~~

There's a very cool characterization of these graphs which is purely combinatorial.

Theorem. (Tay) A graph G with n vertices and $2n-3$ edges is ~~is~~ ~~(infinitesimally)~~ generically rigid $\Leftrightarrow$ G is a union of 3 trees such that every vertex is contained in exactly two of them AND no two nontrivial subtrees span the same vertices.

Proof. We recall the rigidity matrix

$$A = \begin{array}{rc} \begin{matrix}\#\\ \text{Edges} \\ = \\ 2n-3\end{matrix} \left\{ \vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}} \right. & \underbrace{\begin{bmatrix} \cdots & & & & \\ \cdots & p_i - p_j & \cdots & p_j - p_i & \cdots \\ & & & & \end{bmatrix}}_{2 \times \#\text{vertices} = 2n} \\ & \begin{matrix} \uparrow \quad\quad\quad \uparrow \\ \text{position } i \quad \text{position } j \end{matrix} \end{array}$$

The product Av for a vector of velocities $[v_1 \cdots v_{\#\text{Verts}}]^T = v$ gives the derivatives of edgelengths. An embedding is rigid if $\operatorname{rank} A = 2n-3$.

=> ~~If A has rank 2n-3, then it has row rank 2n-3, and we can delete edges until~~

=> If A has rank 2n-3, ~~we can know that the last three columns are in the span of the first 2n-3, so we may assume A' = A - last 3 columns is still~~

the equivalence of determinental and matrix rank implies $\exists$ a $2n-3 \times 2n-3$ submatrix which has nonzero determinant.

Now let's (re?) learn a very cool linear algebra trick. We all know how to expand a determinant by minors along a row or column:

$$\begin{vmatrix} a_{11} & & a_{1n} \\ & & \\ a_{n1} & & a_{nn} \end{vmatrix} = \sum (-1)^{i+1} a_{i1} \begin{vmatrix} a_{12} & a_{1n} \\ a_{n2} & a_{nn} \end{vmatrix}$$

$a_{i2} \cdots a_{in}$ is missing

But you can also expand along a collection of rows (or columns) simultaneously. Given a K-element subset H of $1 \ldots n$, rows we can define for each choice L of a K element subset of $1 \ldots n$, columns, the $K \times K$ minor $M_{H,L} = [a_{ij} \mid i \in H, j \in L]$ and a complementary ~~matrix~~ $n-k \times n-k$ minor

$$M_{H',L'} = [a_{ij} \mid i \notin H, j \notin L]$$

then as before

$$\det A = \sum_{\substack{k \text{ elt} \\ \text{subsets} \\ L}} (-1)^{\left(\sum_{h\in H} h + \sum_{e \in L} e\right)} \det M_{H,L} \det M_{H',L'}.$$

Choosing H to be the $n-1$ odd rows of A^T (columns of A), we get at least one L of rows of A so that

$$\det M_{H,L} \text{ and } \det M_{H',L'}$$

are both nonzero. Genericity allows us to assume that the entries in both matrices are nonzero. Reading across a row of the rigidity matrix, we see

$$\cdots\cdots(p_i - p_j)_x \cdots\cdots (p_j - p_i)_x \cdots\cdots$$

in $M_{H,L}$. We can rescale by $\frac{1}{|(p_i - p_j)_x|}$

so that the row reads

$$\cdots\cdots 1 \cdots\cdots -1 \cdots\cdots \qquad \hat{M}^{*}_{H,L}$$

~~Thus the rescaled matrix~~ $\hat{M}^{*}_{H,L}$ ~~is an (oriented) incidence matrix for the subgraph determined by the selection of edges in given by L.~~

Claim. If $\det \hat{M}^{*}_{H,L} \neq 0$, then the subgraph has no cycles.

Proof. Suppose we had a cycle composed of edges $e_1 \cdots e_k$ (as usual, subscripts are a problem, but we could always renumber if the cycle was something else.

Assigning an orientation to the edges so that the cycle is consistently oriented, (flips sign of det, but can't make it 0).

we see

$$[1 \cdots 1\ 0 \cdots 0] \begin{bmatrix} \cdots 1 \cdots & -1 \cdots \\ \cdots -1 \cdots & 1 \cdots \\ & \end{bmatrix} = [0 \cdots 0]$$

each column has a +1 for the incoming edge and a -1 for the outgoing one

Thus $\hat{M}^{*}_{H,L}$ is not full rank, and det $= 0$

We have learned that there is no cycle ~~sobgraphs~~ given by edges in L, L'. ~~are trees~~. ~~have no cycles~~! ~~Now~~ ~~Therefore they are forests~~ (~~each connected component is a tree~~).

~~Now~~

~~L has (n-1) edges and n-1~~

Now the matrices $\hat{M}$ aren't traditional incidence matrices, but th~~e~~ey still define sobgraphs.

The subgraph given by L has $(n-1)$ edges on n verts, so it is a single tree.

The subgraph given by L' has $(n-2)$ edges on ~~ve~~ n verts, so it is a disjoint union of 2 trees.