Important Examples of Graphs

We are now going to think about some important example graphs. We will work out the spectrum of their graph Laplacian ~~and~~ ~~and/~~or bound their eigenvalues when we can't derive the entire spectrum.

To interpret the eigenvalues, we will need.

Definition. If S is a subset of the vertex set V of G, we define the <u>boundary</u> ∂S to be the set of edges connecting vertices in S to vertices not in S.

We define the <u>isoperimetric ratio</u>

$$\Theta(S) := \frac{|\partial S| \;\leftarrow \text{\# of edges in } \partial S}{|S| \;\leftarrow \text{\# of vertices in } S}.$$

We then have a really useful

Theorem. For any $S \subset V$, we have

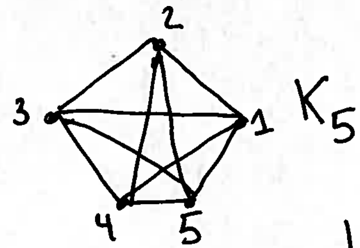
$$\Theta(S) \geq \lambda_2(G)\left(1 - \frac{|S|}{|V|}\right).$$

(We will prove this later in the course.)

Corollary. If d_{max} is the largest degree of any vertex in G, then

$$d_{max} \geq \lambda_2(G)\left(1 - \frac{1}{|V|}\right).$$

Example.



The complete graph K_v has edges joining every pair of vertices, and v vertices total.

We proved in the last notes that the eigenvalues of K_v are

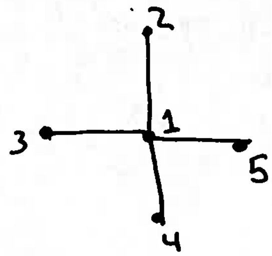
$$\{0, \underbrace{v, v, \ldots, v}_{v-1 \text{ times}}\}.$$

Now $d_{max} = v-1$, so our corollary is

$$v-1 \geq v\left(1-\frac{1}{v}\right) = v\left(\frac{v-1}{v}\right) = v-1.$$

That is, the estimate is sharp for K_v.

Example.



$= S_5$

The star graph S_v has edges $\{1 \bullet\!\!-\!\!\bullet\, i$, for $i \in 2, \ldots, v\}$.

Lemma. If G is a graph, and a and b are vertices of degree 1, and there is some vertex c with $a \bullet\!\!-\!\!\bullet\, c$ and $c \bullet\!\!-\!\!\bullet\, b$, then $\vec{x} \in \mathbb{R}^V$ with $\vec{x} = \delta(a) - \delta(b)$[1] is an eigenvector of L_G with eigenvalue 1.

Proof. Homework.

We can use this to see that S_v has $v-2$ linearly independent eigenvectors

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[1] Remember that $\delta(a)$ is the basis vector with $\vec{x}(a) = 1$ and $\vec{x}(b) = 0$ for all other vertices $b \neq a$.

$\{\delta_2 - \delta_3, \delta_3 - \delta_4, \ldots, \delta_{n-1} - \delta_n\}$[1] all with eigenvalue 1. Since S_v is connected, there is one eigenvector ($\vec{1}$) of eigenvalue 0.

To generate the last eigenvalue, consider

$$\vec{x}(i) = \begin{cases} -(n-1), & \text{if } i = 1 \\ 1, & \text{otherwise.} \end{cases}$$

this is orthogonal to $\vec{1}$ and to each $\delta(a) - \delta(b)$ (with $a, b > 1$). So it <u>must</u> be an eigenvector (slick, right?).

To determine the eigenvalue, we can compute the Rayleigh quotient (also homework!) and get n.

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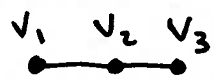
[1] Note. These aren't orthogonal, so they are not the basis for the eigenspace with eigenvalue 1 produced by the spectral theorem. But we could orthogonalize them.

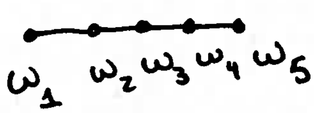
Definition. Let G and H be weighted graphs, with vertex sets V_G, V_H and edge sets E_G, E_H. We define the graph product $G \times H$ to be the graph with

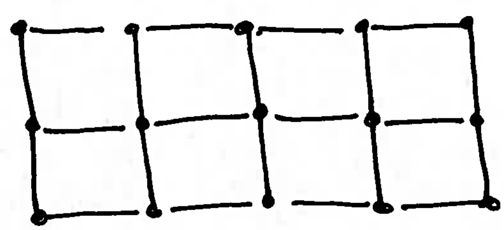
vertex set $V_G \times V_H = \{ (v,w) \mid v \in V_G, w \in V_H \}$

edge set $\left\{ (v,w) \bullet\!\!-\!\!\bullet (\hat{v},\hat{w}) \;\middle|\; \begin{array}{c} (v = \hat{v} \text{ and } w \bullet\!\!-\!\!\bullet \hat{w}) \\ \text{or} \\ (v \bullet\!\!-\!\!\bullet \hat{v} \text{ and } w = \hat{w}) \end{array} \right\}$

Example.

$G =$ 

$H =$ 

 $= G \times H$

Homework: Label all the vertices and edges in the drawing of $G \times H$.

Theorem. Let G, H be weighted graphs with (Laplacian) eigenvalues $\lambda_1, \dots, \lambda_n$ and $\mu_1, \dots, \mu_m$ and eigenvectors $\vec{\alpha}_1, \dots, \vec{\alpha}_n$ and $\vec{\beta}_1, \dots, \vec{\beta}_m$.

For each $1 \leq i \leq n$, & $1 \leq j \leq m$, the product graph $G \times H$ has an eigenvector $\vec{\gamma}_{i,j}$ with eigenvalue $\lambda_i + \mu_j$ so that

$$\vec{\gamma}_{ij} = \vec{\alpha}_i \times \vec{\beta}_j \iff \vec{\gamma}_{ij}(a,b) = \vec{\alpha}(a)\vec{\beta}(b).$$

Proof. Let $\vec{\alpha}$ be an eigenvector of L_G with eigenvalue λ and $\vec{\beta}$ be an eigenvector of L_H with eigenvalue μ. Let

$$\vec{\gamma}(a,b) := \vec{\alpha}(a)\vec{\beta}(b).$$

We claim that $\vec{\gamma}$ is an eigenvector of $L_{G\times H}$ with eigenvalue $\lambda+\mu$. Now

$$\left(L_{G\times H}\,\vec{\gamma}\right)(a,b) = \sum_{\substack{(a,b)\sim(\hat{a},\hat{b}) \\ \text{in } G\times H}} \omega_{(a,b)\sim(\hat{a},\hat{b})}\left(\vec{\gamma}(a,b)-\vec{\gamma}(\hat{a},\hat{b})\right)^{\cancel{2}}$$

so since the edges of $G\times H$ are defined as above, we can rewrite this as

$$= \sum_{\substack{(a,b)\sim(\hat{a},b) \\ \text{where } a\sim\hat{a} \\ \text{in } G}} \omega_{a\sim\hat{a}}\left(\vec{\gamma}(a,b)-\vec{\gamma}(\hat{a},b)\right)^{\cancel{2}}$$

$$+ \sum_{\substack{(a,b)\sim(a,\hat{b}) \\ \text{where } b\sim\hat{b} \\ \text{in } H}} \omega_{b\sim\hat{b}}\left(\vec{\gamma}(a,b)-\vec{\gamma}(a,\hat{b})\right)^{\cancel{2}}$$

$$= \sum_{\substack{(a,b)\leftrightarrow(\hat{a},b) \\ \text{where } a\leftrightarrow\hat{a} \\ \text{in } G}} \omega_{a\leftrightarrow\hat{a}} \left(\vec{\alpha}(a)\vec{\beta}(b) - \vec{\alpha}(\hat{a})\vec{\beta}(b)\right)$$

$$+ \sum_{\substack{(a,b)\leftrightarrow(a,\hat{b}) \\ \text{where } b\leftrightarrow\hat{b} \\ \text{in } H}} \omega_{b\leftrightarrow\hat{b}} \left(\vec{\alpha}(a)\vec{\beta}(b) - \vec{\alpha}(a)\vec{\beta}(\hat{b})\right)$$

$$= \vec{\beta}(b)\left(\sum_{\substack{a\leftrightarrow\hat{a} \\ \text{in } G}} \omega_{a\leftrightarrow\hat{a}} \left(\vec{\alpha}(a) - \vec{\alpha}(\hat{a})\right)\right)$$

$$+ \vec{\alpha}(a)\left(\sum_{\substack{b\leftrightarrow\hat{b} \\ \text{in } H}} \omega_{b\leftrightarrow\hat{b}} \left(\vec{\beta}(b) - \vec{\beta}(\hat{b})\right)\right)$$

$$= \vec{\beta}(b)\left(L_G\vec{\alpha}\right)(a) + \vec{\alpha}(a)\left(L_H\vec{\beta}\right)(b)$$

$$= \lambda\,\vec{\alpha}(a)\vec{\beta}(b) + \mu\,\vec{\alpha}(a)\vec{\beta}(b)$$

$$= (\lambda+\mu)\,\vec{\alpha}(a)\vec{\beta}(b), \text{ as desired.}$$

To see that these are <u>all</u> the eigenvectors and eigenvalues of $L_{G\times H}$ we just count. There are $n\times m$ vertices in $G\times H$, so there are $n\times m$ eigenvectors. We have given $n\times m$ eigenvectors already (n $\vec{\alpha}$'s and m $\vec{\beta}$'s) so we have given them all. $\square$

Example. The hypercube.

We define H_d to be the graph with vertex set $\{0,1\}\times\dots\times\{0,1\}=\{0,1\}^d$ and $(b_1\dots b_d) \leftrightarrow (\hat{b}_1\dots\hat{b}_d)$ if $b_i=\hat{b}_i$ for exactly $(d-1)$ of the digits i.

Fun fact: The distance in H_d between two bit strings is = # of bits where they differ, called Hamming distance

We can compute $H_1 = \underset{0}{\bullet}\!\!\text{———}\!\!\underset{1}{\bullet}$, with $L_{H_1} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. You can check directly that

$$L_{H_1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad L_{H_1}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

so the eigenvalues of H_1 are 0 and 2.

Now we can compute the eigenvalues and vectors of H_d inductively using the fact that $H_d = H_{d-1} \times H_1$.

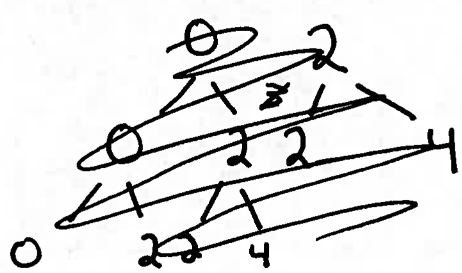
If $\vec{\psi}$ is an eigenvector of H_{d-1} with eigenvalue λ, then

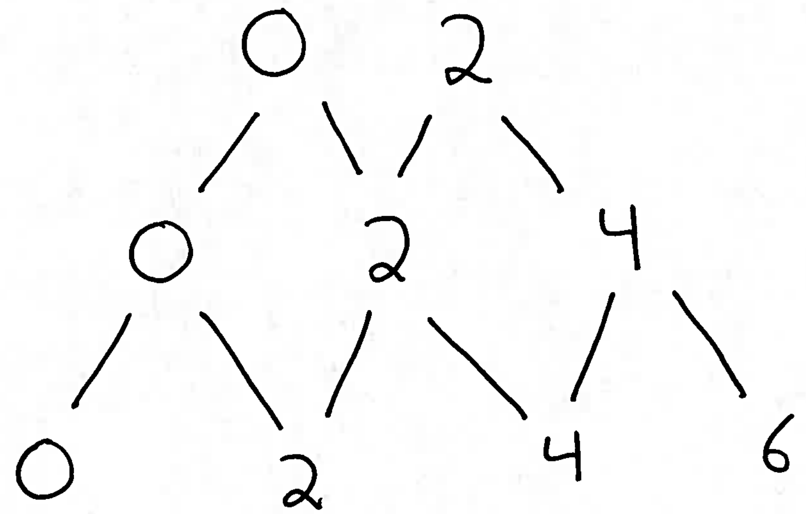
$$\begin{bmatrix} \vec{\psi} \\ \vec{\psi} \end{bmatrix} \left(= \vec{\psi} \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

and

$$\begin{bmatrix} \vec{\psi} \\ -\vec{\psi} \end{bmatrix} \left(= \vec{\psi} \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)$$

are eigenvectors of $H_d = H_{d-1} \times H_1$ with eigenvalues λ and $\lambda + 2$. Thus the eigenvalues of H_d are





or $2i$ with ~~eig~~ multiplicity $\binom{d}{i}$.
The multiplicity follows from the fact that we choose to add 0 or 2 at each level of the tree above and must choose 2 exactly i (of d) times.

The eigenvectors are interesting combinations of (-1)'s and 1's, which we'll work out in homework.

Definition. The isoperimetric ratio of a graph G is given by

$$\Theta(G) = \min_{\substack{S \subset V \\ |S| \leq |S^c|}} \Theta(S)$$

(where $S^c = V - S$).

Using the fact that $\lambda_2(H_d) = 2$, we can now show

$$\Theta(S) \geq 2\left(1 - \frac{|S|}{|V|}\right)$$

so if $|S| \leq |S^c| = |V| - |S|$, we have $\frac{|S|}{|V|} \leq \frac{1}{2}$ and so

$$\Theta(S) \geq 2(1/2) = 1.$$

Thus $\Theta(H_d) \geq 1$.

Example. If we take $S = \{(b_1 \ldots b_d) \mid b_1 = 0\}$ then $|S| = 2^{d-1} = |S^c|$ and $|\partial S| = |S|$ as each $(0\, b_2 \cdots b_d) \longleftrightarrow (1\, b_2 \cdots b_d)$ is the unique edge joining each vertex in S to one in S^c. Thus $\Theta(S) = 1$ and so we have

Proposition. $\Theta(H_d) = 1$.