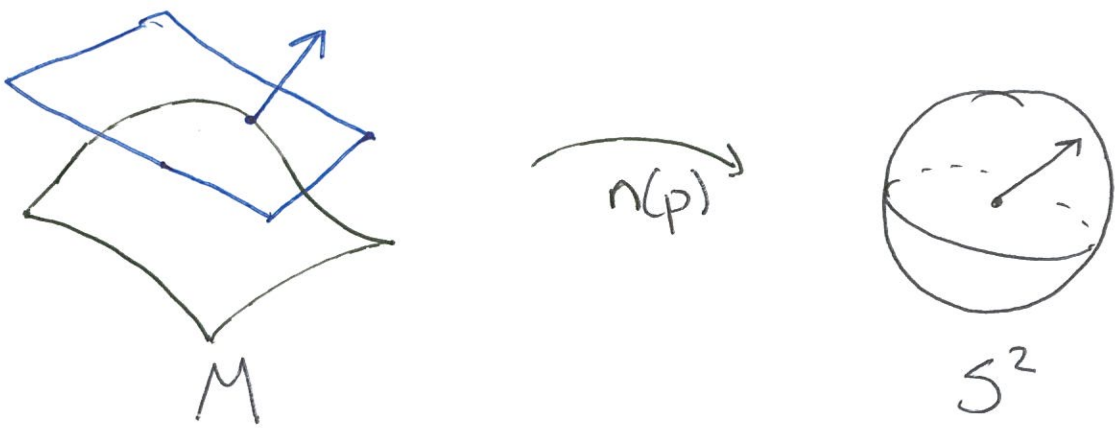
Gauss Map and Second Fundamental Form.

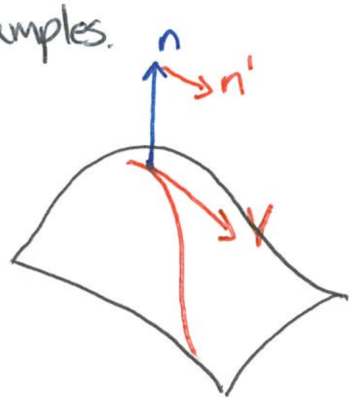
Given a surface, there's a natural map

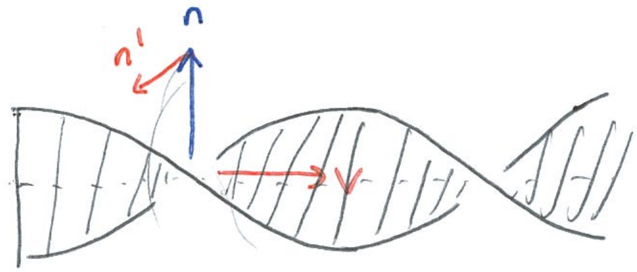


assigning the unit normal $n(p)$ to each point p on M.

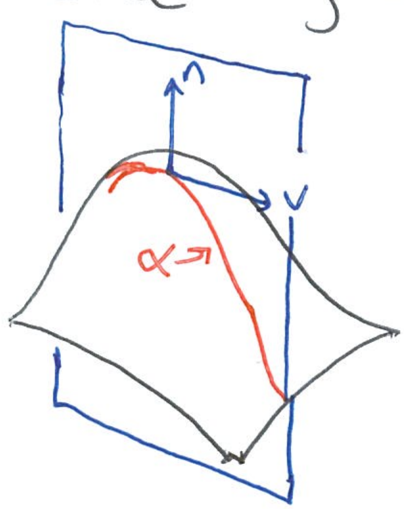
We are going to use $n(p)$ and its derivatives to understand how M curves at P. We can think of $n(p)$ as a kind of tangent indicatrix for surfaces_ the derivative of T yields curvature.

Examples.





The directional derivative $D_v n$ tells us how M is bending in the v direction. In fact, if we slice M by the plane containing $\vec{v}$ and $\vec{n}$, and parametrize the slice by arclength as $\alpha(s)$, then



$\alpha(0) = p$

$\alpha'(0) = \vec{v}$

$N(0) = \pm \vec{n}(p)$

↑ principal normal of α

Claim. $\pm \kappa(0) = -\langle D_v n, v \rangle$.

Since α is in the surface, $\langle \alpha', n(\alpha(s)) \rangle = 0$ or $\langle T(s), \vec{n}(\alpha(s)) \rangle = 0$. So

$$\frac{d}{ds} \langle T(s), \vec{n}(\alpha(s)) \rangle = \langle T'(s), \vec{n}(\alpha(s)) \rangle + \langle T(s), \frac{d}{ds} \vec{n}(\alpha(s)) \rangle$$

$$= \langle T'(s), \vec{n}(\alpha(s)) \rangle + \langle T(s), D_{T(s)} \vec{n} \rangle$$
$$= 0,$$

using the fact that the directional derivative of any function on a surface is the ordinary derivative w.r.t. arclength along a curve with tangent vector equal to the direction. So we can write

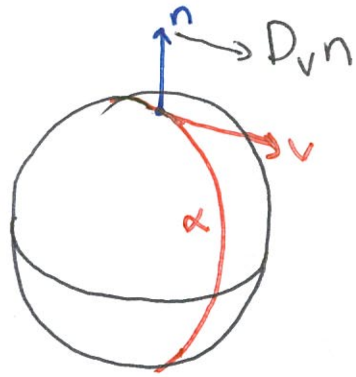
$$-\langle \vec{v}, D_{\vec{v}} \vec{n} \rangle = \langle T', \vec{n} \rangle$$
$$= \langle \kappa N, \vec{n} \rangle$$
$$= \pm \kappa(0). \qquad \square$$

So the, directional derivative in the

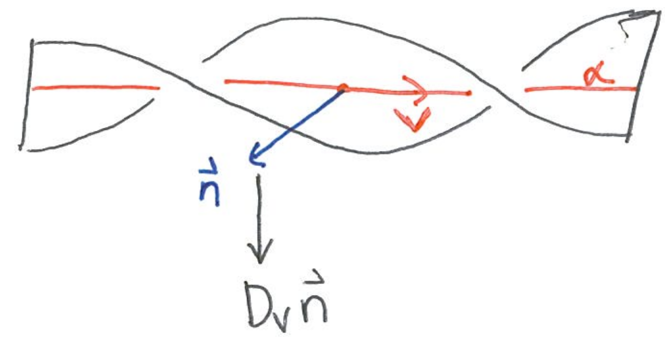
fraction of the

direction of variation tells us the curvature of that slice.

We can check



lots of curvature for α



no curvature for α

We now prove

Proposition. For any $\vec v \in T_pM$, $D_{\vec v}\vec n \in T_pM$.
We can define a symmetric linear map

$$S_p(\vec v) = -D_{\vec v}\vec n$$

called the shape operator.

Proof. Since $\vec n$ has unit norm,

$$0 = D_{\vec v}\langle \vec n, \vec n\rangle = 2\langle D_{\vec v}\vec n, \vec n\rangle$$

so $D_v \vec n$ is in the plane with normal $\vec n$ – that is, T_pM.

Note: That's the first time we've differentiated an inner product with a <u>directional</u> derivative, but remember that

$$\langle \vec{n}(x(u,v)), \vec{n}(x(u,v)) \rangle = g(u,v)$$

is constant ~~in~~ as a function of u,v and $D_v g$ is just a linear combination of u and v partials.

We now show symmetry: for any $\vec{u},\vec{v}$

$$\langle S_p(\vec{u}), \vec{v} \rangle = \langle \vec{u}, S_p(\vec{v}) \rangle.$$

First, suppose $\vec{u} = \vec{X}_u$, $\vec{V} = \vec{X}_v$. Now $\vec{X}_u, \vec{X}_v$ are always tangent vectors, so

$$\langle \vec{n}, X_u \rangle = 0, \Rightarrow \langle \vec{n}_u, \vec{X}_u \rangle = -\langle \vec{n}, \vec{X}_{uu} \rangle$$

$$\langle \vec{n}, X_u \rangle = 0 \Rightarrow \langle \vec{n}_u, \vec{X}_v \rangle = -\langle \vec{n}, \vec{X}_{uv} \rangle.$$

(here $\vec{n}_u = D_{x_u}\vec{n} = \frac{\partial}{\partial u}\vec{n}(x(u,v))$.). So

$$\langle S_p(x_u), x_v\rangle = -\langle D_{x_u}\vec{n}, x_v\rangle$$

$$= -\langle \vec{n}_u, \vec{x}_v\rangle$$

$$= \langle \vec{n}, \vec{x}_{vu}\rangle$$

mixed partials commute!

$$= \langle \vec{n}, \vec{x}_{uv}\rangle$$

$$= -\langle \vec{n}_v, \vec{x}_u\rangle = -\langle D_{x_v}\vec{n}, x_u\rangle = \langle S_p(x_v), x_u\rangle.$$

Now to prove symmetry for any $\vec{u}, \vec{v}$ we just write them as linear combin. of x_u, x_v and expand / simplify.

I'm going to leave this as an exercise, though Shifrin writes it out.

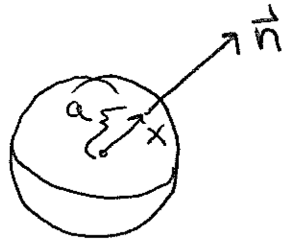
Examples.

If $S_p(\vec{v}) = 0$ for all p, v then M is a plane (or a subset of a plane).

Since $S_p(\vec{v}) = -D_{\vec{v}}\vec{n}$, this shows all directional derivatives of $\vec{n}$ are 0. Hence $\vec{n}$ is constant and M can't leave the plane normal to $\vec{n}$.

If M is a sphere, of radius a, then $S_p(\vec{v}) = -\frac{1}{a}\vec{v}$.

We know that



$\vec{n}$ is a unit vector in the direction of $\vec{x}$, so $\vec{n} = \frac{1}{a}\vec{x}$. Thus $D_{\vec{v}}\vec{n} = D_{\vec{v}}\left(\frac{1}{a}\vec{x}\right) = \frac{1}{a}\vec{v}$. This last takes a little unpacking, but recall that x is a function of u and v (the parameters) so

$$\vec{n}(u,v) = \frac{1}{a}\vec{X}(u,v) \Rightarrow \begin{array}{l} \vec{n}_u = \frac{1}{a}\vec{X}_u \\ \vec{n}_v = \frac{1}{a}\vec{X}_v \end{array}$$

Thus if $\vec{V} = \lambda_1 \vec{X}_u + \lambda_2 \vec{X}_v$ we have

$$D_{\vec{V}}\vec{n} = \lambda_1 n_u + \lambda_2 n_v =$$

$$= \lambda_1/_a \, X_u + \lambda_2/_a \, X_v$$

$$= \frac{1}{a}(\lambda_1 X_u + \lambda_2 X_v) = \frac{1}{a}\vec{V}$$

as claimed. Thus $S_p(\vec{v}) = -D_{\vec{v}} n = -\frac{1}{a}\vec{v}$.