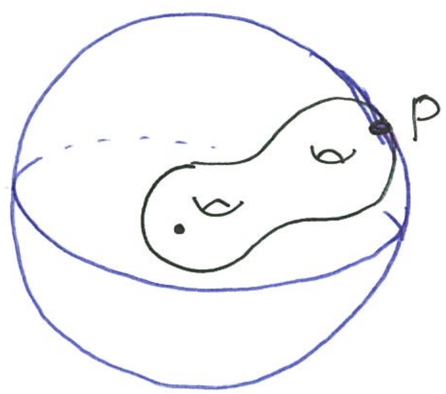
Global Geometry of Surfaces.

We start with something easy:

Proposition. Suppose $M \subset \mathbb{R}^3$ is compact. Then $\exists$ $p \in M$ so that $K(p) > 0$.

Proof.



Some p on M is farthest from $\vec{0}$ by compactness. At this point, the normal $\vec{n}(p)$ is colinear with $\vec{p}$, so M and the sphere of radius $\|\vec{p}\|$ share a tangent plane.

Now any curve $\alpha(s)$ on M has

$$\frac{d}{ds}\langle \alpha(s), \alpha(s) \rangle = 0, \quad \frac{d^2}{ds^2}\langle \alpha(s), \alpha(s) \rangle \leq 0.$$

So

$$\langle T(s), \alpha(s) \rangle = 0, \quad \langle \kappa N(s), \alpha(s) \rangle + \langle T(s), T(s) \rangle \leq 0$$

or

$$\langle KN(s), \alpha(s) \rangle \le -1$$

This means

$$|\langle KN(s), \alpha(s) \rangle| \ge 1$$

or

$$K \overset{1}{|\cancel{N}|}\, |\alpha(s)|\, \overset{\le 1}{\cancel{|\cos\theta|}} \ge 1.$$

Since $|\cos\theta| \le 1$, this means

$$K \ge 1/|\alpha(s)|.$$

Thus the principal curvatures of M are each at least $1/|p|$ and the curvature of M is $\ge 1/|p|^2$. $\square$

Now we can prove something cool! We've already seen ~~that~~ a compact surface of constant positive Gauss curvature (the sphere) and ~~of~~ a noncompact surface of constant negative Gauss curvature (the pseudosphere).

Can we fix up the pseudosphere to be compact and smooth?

Theorem. If M is a smooth compact surface of constant curvature K, then $K>0$ and M is a sphere.

We need a Lemma.

Lemma. Suppose p is not umbilic, and $K_1(p) > K_2(p)$. If K_1 has a local max at p and K_2 a local min, then $K(p) \leq 0$.

Proof. We invoke our principal parametrization, where u-curves are lines of K_1-curvature, and v-curves are lines of K_2-curvature.

Now ~~K_1~~ remember that

$$(K_1)_v = \frac{E_v}{2E}(K_2 - K_1) \quad \text{and} \quad (K_2)_u = \frac{G_u}{2G}(K_1 - K_2)$$

Since $K_1 \neq K_2$ and ~~a~~ we are at a critical point for K_1 and K_2, we can conclude

$$E_v = G_u = 0.$$

Taking another derivative,

$$(K_1)_{vv} = \left(\frac{E_{vv} 2E - \cancelto{0}{E_v 2E_v}}{(2E)^2} \right)(K_2 - K_1)$$

$$+ \cancelto{0}{\frac{E_v}{2E}(K_2 - K_1)_v}$$

$$= \frac{E_{vv}}{2E}(K_2 - K_1) \leq 0 \quad \text{(since } K_1 \text{ has a local } \underline{\text{max}} \text{ at } p\text{)}$$

and

$$(K_2)_{uu} = \left(\frac{G_{uu} 2G - \cancelto{0}{G_u 2G_u}}{(2G)^2} \right)(K_1 - K_2)$$

$$+ \cancelto{0}{\frac{G_u}{2G}(K_1 - K_2)_u}$$

$$= \frac{G_{uu}}{2G}(K_1 - K_2) \geq 0 \quad \text{(since } K_2 \text{ has a local } \underline{\text{min}} \text{ at } p\text{)}$$

Now $E = \langle x_u, x_u \rangle$ and $G = \langle x_v, x_v \rangle$ are always ≥ 0, and we know $K_1 > K_2$, so this implies that

$$E_{vv} \geq 0 \quad \text{and} \quad G_{uu} \geq 0.$$

But now since $F = 0$, we have

$$K = -\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right)$$

(we'll prove this in homework) so

$$= \frac{-1}{2\sqrt{EG}}\left(\frac{E_{vv}}{\sqrt{EG}} - \frac{\cancelto{0}{E_v}\,\frac{1}{2}(EG)^{-1/2}(E_vG + EG_v)}{(\sqrt{EG})^2}\right.$$

$$\left. + \frac{G_{uu}}{\sqrt{EG}} - \cancelto{0}{G_u}\,(\text{stuff})\right)$$

$$= \frac{-1}{2EG}(E_{vv} + G_{uu}) \leq 0, \quad \text{as desired. } \square$$

Now we can do the proof of the main theorem.

We already know $K > 0$, because M has a point where $K > 0$ by the lemma earlier.

Now the larger principal curvature κ_1 is certainly continuous, so it reaches a absolute max at some p.

Since $\kappa_1 \kappa_2$ is constant, this point is an absolute <u>min</u> of κ_2.

Case 1. κ_1, κ_2 are different.

By the Lemma, $K(p) \leq 0$.

Case 2. $\kappa_1 \kappa_2$ are the same.

Since $\kappa_1 \geq \kappa_2$ by definition, ~~and~~ (κ_1 was the <u>larger</u> principal curvature), this means $\kappa_1 = \kappa_2$ everywhere (all points are umbilic!)

You'll prove in homework that this means M is a sphere. $\square$