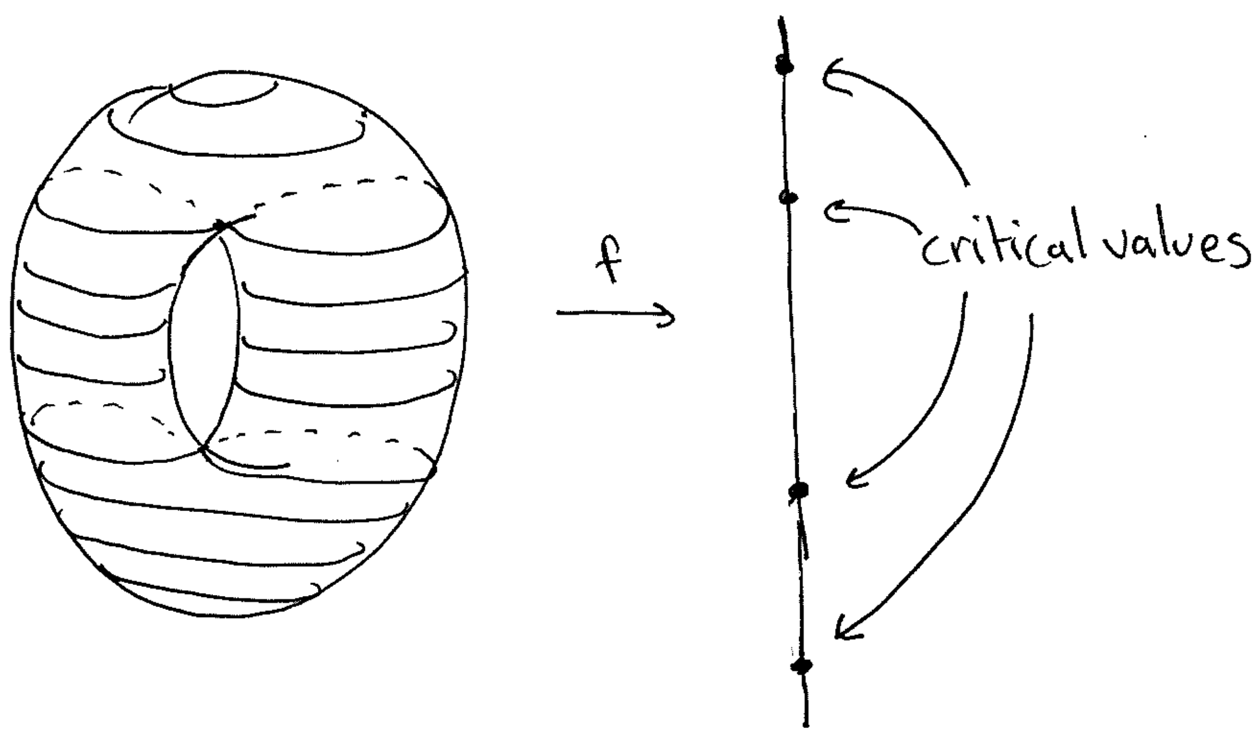
Review of Morse Theory

pages numbered 5 - 14.

We will shortly introduce an explicit cell decomposition on these manifolds which makes the cohomology ring clear (or at least calcolatable!). But for now we will give an approach via Morse-Bott theory.

First, we recall the setup of classical Morse theory. Given a manifold M, we consider functions $f: M \to \mathbb{R}$.



The basic idea is that we can construct a particular CW complex structure on M by following trajectories of ∇f to critical points. We can then compute cellular homology using this CW structure. We need a technical hypothesis:

Definition. Let p_0 be a critical point of the smooth function $f: M \to \mathbb{R}$. The Hessian of f at p_0 is a quadratic form on $T_{p_0}M$ given by

$$\mathrm{Hess}_{f,p_0}(\vec{v},\vec{w}) = XYf(p_0),$$

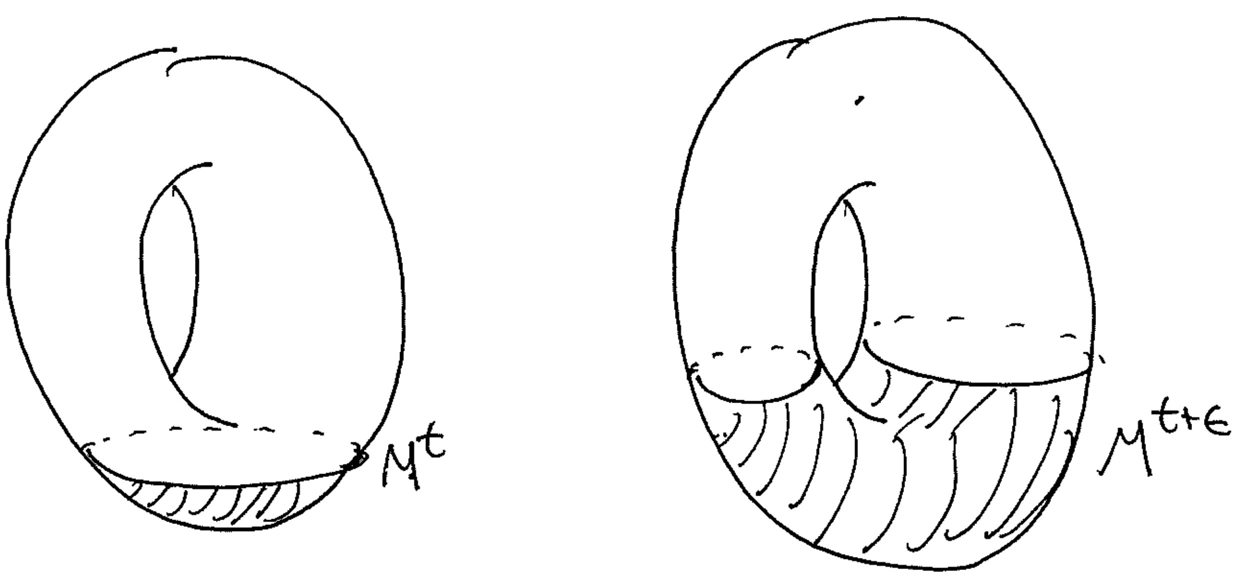
where X and Y are any smooth vector fields extending $\vec{v}, \vec{w}$ and Xf is the derivative of f along the orbit of X as usual.

The Hessian is <u>non-degenerate</u> if

$$\mathrm{Hess}_{f,p_0}(\vec{v},\vec{\omega})=0 \text{ for all } \vec{\omega} \Rightarrow \vec{v}=0.$$

A smooth function is a Morse function if ~~not~~ the Hessian is nondegenerate at each critical point.

The <u>index</u> of a nondegenerate critical point is the number of negative eigenvalues of the Hessian.



Consider the sublevel sets

$$M^t = \{p \in M \mid f(p) \leq t\}.$$

We see that the topology of M^t changes only when t passes a critical point.

Theorem (weak and strong Morse principles)

If $f^{-1}[a,b]$ contains no critical points, then M^a is diffeomorphic to M^b and M^a is a deformation retract of M^b.

If $f^{-1}[a,b]$ contains a single nondegenerate critical point of index λ, then M^b has the homotopy type of M^a with a λ-cell attached.

In fact, if $f^{-1}[a,b]$ contains K nondegenerate critical points of indices $\lambda_1, \ldots, \lambda_K$, M^b has the homotopy type of M^a with K cells of dimensions $\lambda_1, \ldots, \lambda_K$ attached.

We now do an explicit example. Consider $\mathbb{C}P^n$, which is $\mathbb{C}^{n+1}/\mathbb{C}$ where the $\mathbb{C}$ acts by multiplying each coordinate. This is the complex Grassmannian $G_1(\mathbb{C}^n)$, which we write as ratios $z_0 : \ldots : z_n$ s.t. $\sum |z_i|^2 = 1$. Consider the coordinate chart on

$$U_i = \{ z_0 : \ldots : z_n \mid z_i \neq 0 \}$$

given by $\varphi_i(z_0 : \ldots : z_n) = |z_i| \left(\frac{z_0}{z_i}, \ldots, \frac{\hat{z_i}}{z_i}, \ldots, \frac{z_n}{z_i} \right)$

(the hat means this term is missing, as usual).

Now take the function

$$f(z_0 : \ldots : z_n) = \sum c_i |z_i|^2, \qquad c_i \neq c_j \neq 0.$$

We want to write this in the local coordinates given by φ_0. Now let $x_j + i y_j = |z_i| \frac{z_j}{z_i}$. We have

$$x_j^2 + y_j^2 = (x_j + i y_j)(x_j - i y_j)$$
$$= |z_i| \frac{z_j}{z_i} \, |z_i| \frac{\overline{z_j}}{\overline{z_i}} = \frac{|z_i|^2}{z_i \overline{z_i}} z_j \overline{z_j} = |z_j|^2.$$

We now have

$$|z_0|^2 = \sum_{j=0}^{n} |z_j|^2 - \sum_{j=1}^{n} |z_j|^2$$

$$= 1 - \sum_{j=1}^{n} |z_j|^2$$

$$= 1 - \sum_{j=1}^{n} x_j^2 + y_j^2 .$$

So in these coordinates,

$$f = c_0 |z_0|^2 + \sum_{i=1}^{n} c_i |z_i|^2$$

$$= c_0 \left(1 - \sum_{j=1}^{n} x_j^2 + y_j^2\right) + \sum_{j=1}^{n} c_j (x_j^2 + y_j^2)$$

$$= c_0 + \sum_{j=1}^{n} (c_j - c_0)(x_j^2 + y_j^2).$$

This is critical $\Leftrightarrow$ $x_j = y_j = 0$ for $j \in 1, \dots, n$, or at the point $1:0:\dots:0$. The index of the critical point is equal to twice the number of c_j for which $c_j < c_0$.

This ~~a~~ argument works for each coordinate neighborhood φ_i, revealing critical points at $0:\ldots:1:0:\ldots:0$ for each i.
↑ ith position

The index of this critical point is

$$2(\#\text{ of } j \text{ for which } c_j - c_i < 0)$$

Clearly, while this index depends on the ordering of the c_i, over the complete set of $n+1$ critical points, each of

$$0, 2, 4, \ldots, 2n$$

occurs exactly once.

This means that $\mathbb{C}P^n$ can be constructed with one cell in each even dimension, and so has homology

$$H_i(\mathbb{C}P^n; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i \text{ even}, 0 \leq i \leq 2n \\ 0, & \text{otherwise.} \end{cases}$$

The Morse Inequalities.

The basic idea of the Morse inequalities is simple: if critical points are cells, and homology requires cells, then homology requires critical points.

But there's actually more to say. First, define the ring of (formal) Laurent series:

$$\mathbb{Z}[t,t^{-1}] = \left\{ \sum_{n\in\mathbb{Z}} a_n t^n \,\middle|\, \begin{array}{l} a_n \in \mathbb{Z},\ \exists N>0, \text{ so} \\ a_n = 0 \text{ for } n < -N. \end{array} \right\}.$$

We can define an order relation by

$$X(t) \succ Y(t) \iff Q(t) \text{ with nonnegative coefficients so that } X(t) = Y(t) + (1+t)Q(t).$$

Definition. The <u>Morse polynomial</u> $P_f(t)$ associated to a Morse function with critical points $p_1, \ldots, p_k$. ~~~~

is given by

$$P_f(t) = \sum_{\lambda \geq 0} C_\lambda t^\lambda$$

where $C_\lambda = \#$ of critical points of index λ.

Definition. Let $B_\lambda(X)$ be the λ'th Betti number of X (the dimension of $H_\lambda(X;\mathbb{R})$). The Poincaré polynomial $P_X(t)$ is given by

$$P_X(t) = \sum_\lambda B_\lambda(X)\, t^\lambda.$$

We then have

Theorem (Topological Morse Inequalities)
If $f: M \to \mathbb{R}$ is a Morse function on a smooth compact manifold M, then

$$P_f(t) > P_M(t)$$

and in particular $P_f(-1) = P_M(-1) = \chi(M)$.

Corollary. $B_\lambda(M) \le C_\lambda$ for all λ.

Definition. If $P_f(t) = P_M(t)$, then f is called a <u>perfect</u> Morse function.

Examples. In our ~~decompo~~ example of $\mathbb{C}P^n$,

$$P_f(t) = t^{2n} + t^{2n-2} + \dots + t^2 + 1,$$

but by our computation of the Betti numbers,

$$P_M(t) = t^{2n} + t^{2n-2} + \dots + t^2 + 1$$

and f was perfect.