Schubert Conditions and Intersections.

We start by recalling our basic setup. The Plucker embedding maps

$$G_K(\mathbb{C}^n) \to \mathbb{P}^N, \text{ where } N = \binom{K}{n} - 1$$

by taking determinants of submatrices of the $K \times n$ matrix whose rows determine a given K-plane.

These determinants obey Plücker relations given by any (K-1)-moltindex $j = j_1 \cdots j_{K-1}$ and ~~k~~ = (K+1)-moltindex $l = l_1 \cdots l_{K+1}$.

$$\sum_{\lambda=1}^{K} (-1)^\lambda p(j_1 \cdots j_{K-1} l_\lambda) p(l_1 \cdots \hat{l}_\lambda \cdots l_{K+1}) = 0.$$

Last, we can see a K-plane in $\mathbb{C}^n$ as a K-1 plane in $\mathbb{C}P^{n-1}$ by taking intersections with the $\mathbb{C}^{n-1}$ at $(1, \ldots\ldots)$.

We now want to work out conditions for a given $(K-1)$-plane in $\mathbb{C}P^{n-1}$ to intersect a flag of subspaces in a given way.

Let

$$A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_{K-1}$$

~~by~~ be a chain of linear subspaces of $\mathbb{C}P^n$.
We say a $(K-1)$-subspace L <u>obeys the Schubert condition</u> ~~if~~ <u>given by $A_0, \ldots, A_{K-1}$</u> if

$$\dim(A_i \cap L) \geq i$$

for all i.

~~Example:~~
Definition. $\Omega(A_0, \ldots, A_{K-1})$ is the set of $(K-1)$ dimensional spaces obeying the Schubert condition $(A_0, \ldots, A_{K-1})$.

Example. Let A_0 be a line in $\mathbb{C}P^3$ and A_1 be $\mathbb{C}P^3$. Then A_0, A_1 is a flag and $\Omega(A_0, A_1)$ a set of lines in $\mathbb{C}P^3$.

Since every line intersects $\mathbb{C}P^3$ itself in a 1-d subspace,

$$\Omega(A_0, A_1) = \{\text{lines intersecting } A_0\}.$$

Proposition. Let $0 \leq a_0 < \cdots < a_{k-1} \leq n-1$, and for $i = 0, .., k-1$, let A_i be the a_i dimensional linear subspace in $\mathbb{C}P^{n-1}$ of points in the form

$$A_i = \{(x_1, \ldots, x_{a_i}, 0, .., 0)\}$$

Then $\Omega(A_0, .., A_{k-1})$ consists of exactly the points in $\mathbb{C}P^N$ where the determinant $p(j_0, \ldots, j_{k-1}) = 0$ whenever $j_i > a_i$ for some i.

Proof.

Suppose $L \subset \Omega(A_0, \dots, A_{K-1})$. Choose K points $P_0, \dots, P_{K-1}$ so that $P_i \subset A_i \cap L$ and the P_i are linearly independent. Then the P_i span L.

By construction, since

$$L = \begin{bmatrix} p_0(0) & & \cdots & p_0(n-1) \\ p_1(0) & p_1(1) & \cdots & p_1(n-1) \\ \vdots & & & \\ p_{K-1}(0) & p_{K-1}(1) & \cdots & p_{K-1}(n-1) \end{bmatrix}$$

the Plucker coordinate $j = (j_0, \dots, j_{K-1})$ is given by

$$\det \begin{bmatrix} \uparrow \\ p_i(j_B) \\ \downarrow \end{bmatrix}$$

Now suppose for some λ, $j_\lambda > a_\lambda$. Then we have a block of zeros in the upper-right corner of the matrix.

We now see how large the block is by writing

$$L = \begin{array}{c} \left[\begin{array}{ccccccc} p_0(0) & \cdots & p_0(a_0) & 0 & & \vdots & \\ p_1(0) & & \cdots & p_1(a_1) & & 0 & \\ & & & & & \vdots & \\ p_\lambda(0) & & & & p_\lambda(a_\lambda) & 0 & 0 \cdots 0 \\ & & & & & \vdots & \\ p_{K-1}(0) & & & & & & p_{K-1}(a_{K-1}) \quad 0 \end{array}\right] \\ \begin{array}{ccccccc} & & & & & \hphantom{0} j_\lambda \text{ column} & \end{array} \end{array}$$

so

$$L_j = \left.\left[\begin{array}{c|c} \quad & \overbrace{\quad 0 \quad}^{K-\lambda} \\ \hline \quad & \quad \end{array}\right]\right\}\lambda+1$$

where L_j is a ~~$(K-1)\times K$~~ $K\times K$ matrix.

We claim that such a determinant is always zero.

To prove it, we need a neat linear algebra trick!

We all Know how to expand along a row ~~of~~ or column by minors. In fact, we can expand along a collection of rows or columns at the same time!

Let A be an $n \times n$ matrix, and $r = (r_1, \ldots, r_k)$ be a k-multindex of row numbers while $c = (c_1, \ldots, c_k)$ is a multindex of column numbers.

$S(A; r, c)$ = submatrix formed by <u>Keeping</u> $a_{ij} \Leftrightarrow i \in r, j \in c$

$S'(A; r, c)$ = submatrix formed by <u>deleting</u> all rows in r <u>and</u> all columns in c

Laplace Expansion Theorem.

We have for any fixed r,

$$\det A = (-1)^{|r|} \sum_c (-1)^{|c|} \det(S(A,r,c)) \det(S'(A,r,c)).$$

or for any fixed c,

$$\det A = (-1)^{|c|} \sum_r (-1)^{|r|} \det(S(A,r,c)) \det(S'(A,r,c))$$

where $|r| = \sum r_i$ and $|c| = \sum c_i$.

We now set $c = (K-\lambda, \ldots, K)$ and consider

$$\det L_j = (-1)^{|c|} \sum_r (-1)^{|r|} \det(S(L_j,r,c)) \det(S'(A,r,c)).$$

So consider

$$S(L_j;r,c) = \begin{array}{l}\text{a selection}\\ \text{of } K-\lambda \text{ rows}\\ \text{from the}\\ \text{matrix}\\ K\times(K-\lambda)\end{array} \left[\begin{array}{c} 0 \\ \hline * \end{array}\right] \begin{array}{l} \}\ \lambda+1 \\ \}\ K-\lambda-1. \end{array}$$

since this must include a zero row, we're done.

Schubert Relations II.

We have now proved that

$$L \in \Omega(A_0, \ldots, A_{K-1}) \Rightarrow p(j_0, \ldots, j_{K-1}) = 0 \text{ whenever } j_\lambda > a_\lambda \text{ for some } \lambda.$$

We need to go the other way. So suppose we have some point L in the Plucker embedding of $G_K(\mathbb{C}^n) \subset \mathbb{C}P^N$ so that $p(j_0, \ldots, j_{K-1}) = 0$ whenever $j_\lambda > a_\lambda$ for some λ.

Among the nonzero coordinates of L, there is some $\ell = (\ell_0, \ldots, \ell_{K-1})$ so that $p(\ell) \neq 0$ and $\sum \ell_i$ is maximized. We can rescale the coordinates of L so that $p(\ell_0, \ldots, \ell_{K-1}) = 1$.

Now in our construction of ~~pt~~ a K-plane from a point in $\mathbb{C}P^N$ satisfying the

Plucker relations, we saw that after such a rescaling, we could construct a basis for L by the matrix

$$L = \left[\quad \right] \qquad L_{ij} = p(l_0, \ldots, l_{i-1}, j, l_{i+1}, \ldots, l_{K-1}).$$

Notice first that if $j > a_i$, we have $L_{ij} = 0$. To see this recall that

$$p(l_0, \ldots, l_{K-1}) \neq 0, \text{ so } l_i \leq a_i \text{ for all } i$$

thus if $j > a_i$,

$$l_0 + \cdots + l_{i-1} + j + l_{i+1} + \cdots + l_{K-1} > l_0 + \cdots + l_{K-1},$$

and hence $p(l_0, \ldots, l_{i-1}, j, l_{i+1}, \ldots, l_{K-1}) = 0$ because $(l_0, \ldots, l_{K-1}) = l$ was the nonzero coordinate of maximal sum.

Thus the ith row P_i of L lies in A_i, as required for L to satisfy the Schubert conditions $\Omega(A_0, \ldots, A_{K-1})$. $\square$

We now observe that (as expected) the sequence of <u>dimensions</u> $a_0, \dots, a_{k-1}$ of the subspaces of the flag is enough to determine the flag up to a linear transformation.

Proposition. Let $A_0 \subsetneq A_1 \subsetneq \dots \subsetneq A_{k-1}$ and $B_0 \subsetneq B_1 \subsetneq \dots \subsetneq B_{k-1}$ be two strictly increasing sequences of linear subspaces in $\mathbb{C}P^n$ so that $\dim B_i = \dim A_i$ for each i.

Then $\exists$ an invertible linear transformation of $\mathbb{C}^{N+1} \to \mathbb{C}^{N+1}$ which takes the Plucker embedding of $G_k(\mathbb{C}^n)$ to itself and

$$\Omega(B_0, \dots, B_{k-1}) \text{ to } \Omega(A_0, \dots, A_{k-1}).$$

Corollary. For any increasing sequence of subspaces $B_0 \subsetneq B_1 \subsetneq \cdots \subsetneq B_{K-1}$ of $\mathbb{C}P^n$, $\Omega(B_0, \dots, B_{K-1})$ is the intersection of a linear subspace of $\mathbb{C}P^N$ with the Plucker embedding of $G_K(\mathbb{C}^n)$.

The linear space is a hyperplane $\Leftrightarrow$

$$\dim B_0 = n-K, \ \dim B_1 = n-K+2, \ \dim B_2 = n-K+3, \ \dots, \ \dim B_{K-1} = n.$$

Suppose we have $a_0 = n-K, a_1 = n-K+2, \dots, a_{K-1} = n$.

The linear conditions which define $\Omega(A_0, \dots, A_{K-1})$ are that $p(j_0, \dots, j_{K-1}) = 0$ if $\exists \lambda$ so that $j_\lambda > a_\lambda$.

There is only one such ~~sequence~~ multindex j in exactly this case, and it is

$$j_0 = n-K+1, \ j_1 = n-K+2, \ \dots, \ j_{K-1} = n.$$

Example.

We want to know how many lines intersect 4 lines in $\mathbb{RP}^3$, L_1, L_2, L_3, L_4.
~~These~~ lines in $\mathbb{RP}^3$ are represented by the Grassmannian $G_2(\mathbb{R}^4)$, which consists of the points in $\mathbb{RP}^{\binom{4}{2}-1} = \mathbb{RP}^5 \subset \mathbb{R}^6$ which satisfy the single Plücker relation

$$p(12)p(34) - p(13)p(24) + p(14)p(23) = 0.$$

Now we know that the lines intersecting a given line A are given by

$$\Omega(A, \mathbb{RP}^3) \subset G_2(\mathbb{R}^4) \subset \mathbb{RP}^5$$

so the lines intersecting $L_1, .., L_4$ are given by

$$\Omega(L_1, \mathbb{RP}^3) \cap \Omega(L_2, \mathbb{RP}^3) \cap \dots \cap \Omega(L_4, \mathbb{RP}^3).$$

Now the sequence of dimensions for these flags is 1, 3 so we know by the corollary that

$$\Omega(L_i, \mathbb{RP}^3) = G_2(\mathbb{R}^4) \cap H_i$$

where H_i is a hyperplane in $\mathbb{RP}^5$. The intersection of all 4 of these hyperplanes is a line in $\mathbb{RP}^5$, spanned by vectors v, w in $\mathbb{R}^6$.

The intersection of this line with the Grassmannian $G_2(\mathbb{R}^4)$ is given by solutions of the homogenous quadratic given by the Plucker relation:

$$(v_{12}x + w_{12}y)(v_{34}x + w_{34}y)$$
$$-\,(v_{13}x + w_{13}y)(v_{24}x + w_{24}y)$$
$$(v_{14}x + w_{14}y)(v_{23}x + w_{23}y) = 0.$$

This is given by a quadratic form in the vector $\binom{x}{y}$, which either ~~has two solutions~~ has:

1) 2 (orthogonal) lines as solutions
2) 1 line, ~~as~~ "double covered" as solution
3) All (x,y) as solutions.

The first case is the generic one! Therefore, we have learned that:

$$\boxed{\text{2 lines intersect a generic quadruple of lines in } \mathbb{RP}^3}$$