(a) Show that $(fg)^*(\alpha) = f^*(\alpha) + g^*(\alpha)$ for primitive elements $\alpha \in H^*(X;R)$.

(b) Deduce that the $k^{th}$-power map $x \mapsto x^k$ induces the map $\alpha \mapsto k\alpha$ on primitive elements $\alpha$. In particular the quaternionic $k^{th}$-power map $S^3 \to S^3$ has degree $k$.

(c) Show that every polynomial $a_n x^n b_n + \cdots + a_1 x b_1 + a_0$ of nonzero degree with coefficients in $\mathbb{H}$ has a root in $\mathbb{H}$. [See Theorem 1.8.]

**11.** If $T^n$ is the $n$-dimensional torus, the product of $n$ circles, show that the Pontryagin ring $H_*(T^n;\mathbb{Z})$ is the exterior algebra $\Lambda_{\mathbb{Z}}[x_1, \cdots, x_n]$ with $|x_i| = 1$.

**12.** Compute the Pontryagin product structure in $H_*(L;\mathbb{Z}_p)$ where $L$ is an infinite-dimensional lens space $S^\infty/\mathbb{Z}_p$, for $p$ an odd prime, using the coproduct in $H^*(L;\mathbb{Z}_p)$.

**13.** Verify that the Hopf algebras $\Lambda_R[\alpha]$ and $\mathbb{Z}_p[\alpha]/(\alpha^p)$ are self-dual.

**14.** Show that the coproduct in the Hopf algebra $H_*(X;R)$ dual to $H^*(X;R)$ is induced by the diagonal map $X \to X \times X$, $x \mapsto (x,x)$.

**15.** Suppose that $X$ is a path-connected H-space such that $H^*(X;\mathbb{Z})$ is free and finitely generated in each dimension, and $H^*(X;\mathbb{Q})$ is a polynomial ring $\mathbb{Q}[\alpha]$. Show that the Pontryagin ring $H_*(X;\mathbb{Z})$ is commutative and associative, with a structure uniquely determined by the ring $H^*(X;\mathbb{Z})$.

**16.** Classify algebraically the Hopf algebras $A$ over $\mathbb{Z}$ such that $A^n$ is free for each $n$ and $A \otimes \mathbb{Q} \approx \mathbb{Q}[\alpha]$. In particular, determine which Hopf algebras $A \otimes \mathbb{Z}_p$ arise from such $A$'s.

# 3.D The Cohomology of SO(n)

After the general discussion of homological and cohomological properties of H-spaces in the preceding section, we turn now to a family of quite interesting and subtle examples, the orthogonal groups $O(n)$. We will compute their homology and cohomology by constructing very nice CW structures on them, and the results illustrate the general structure theorems of the last section quite well. After dealing with the orthogonal groups we then describe the straightforward generalization to Stiefel manifolds, which are also fairly basic objects in algebraic and geometric topology.

The orthogonal group $O(n)$ can be defined as the group of isometries of $\mathbb{R}^n$ fixing the origin. Equivalently, this is the group of $n \times n$ matrices $A$ with entries in $\mathbb{R}$ such that $AA^t = I$, where $A^t$ is the transpose of $A$. From this viewpoint, $O(n)$ is topologized as a subspace of $\mathbb{R}^{n^2}$, with coordinates the $n^2$ entries of an $n \times n$ matrix. Since the columns of a matrix in $O(n)$ are unit vectors, $O(n)$ can also be regarded as a subspace of the product of $n$ copies of $S^{n-1}$. It is a closed subspace since the conditions that columns be orthogonal are defined by polynomial equations. Hence

$O(n)$ is compact. The map $O(n)\times O(n)\to O(n)$ given by matrix multiplication is continuous since it is defined by polynomials. The inversion map $A \mapsto A^{-1} = A^t$ is clearly continuous, so $O(n)$ is a topological group, and in particular an H–space.

The determinant map $O(n)\to\{\pm 1\}$ is a surjective homomorphism, so its kernel $SO(n)$, the 'special orthogonal group,' is a subgroup of index two. The two cosets $SO(n)$ and $O(n)-SO(n)$ are homeomorphic to each other since for fixed $B\in O(n)$ of determinant $-1$, the maps $A\mapsto AB$ and $A\mapsto AB^{-1}$ are inverse homeomorphisms between these two cosets. The subgroup $SO(n)$ is a union of components of $O(n)$ since the image of the map $O(n)\to\{\pm 1\}$ is discrete. In fact, $SO(n)$ is path-connected since by linear algebra, each $A\in SO(n)$ is a rotation, a composition of rotations in a family of orthogonal 2-dimensional subspaces of $\mathbb{R}^n$, with the identity map on the subspace orthogonal to all these planes, and such a rotation can obviously be joined to the identity by a path of rotations of the same planes through decreasing angles. Another reason why $SO(n)$ is connected is that it has a CW structure with a single 0-cell, as we show in Proposition 3D.1. An exercise at the end of the section is to show that a topological group with a finite-dimensional CW structure is an orientable manifold, so $SO(n)$ is a closed orientable manifold. From the CW structure it follows that its dimension is $n(n-1)/2$. These facts can also be proved using fiber bundles.

The group $O(n)$ is a subgroup of $GL_n(\mathbb{R})$, the 'general linear group' of all invertible $n\times n$ matrices with entries in $\mathbb{R}$, discussed near the beginning of §3.C. The Gram–Schmidt orthogonalization process applied to the columns of matrices in $GL_n(\mathbb{R})$ provides a retraction $r: GL_n(\mathbb{R})\to O(n)$, continuity of $r$ being evident from the explicit formulas for the Gram–Schmidt process. By inserting appropriate scalar factors into these formulas it is easy to see that $O(n)$ is in fact a deformation retract of $GL_n(\mathbb{R})$. Using a bit more linear algebra, namely the polar decomposition, it is possible to show that $GL_n(\mathbb{R})$ is actually homeomorphic to $O(n)\times\mathbb{R}^k$ for $k=n(n+1)/2$.

The topological structure of $SO(n)$ for small values of $n$ can be described in terms of more familiar spaces:

- $SO(1)$ is a point.
- $SO(2)$, the rotations of $\mathbb{R}^2$, is both homeomorphic and isomorphic as a group to $S^1$, thought of as the unit complex numbers.
- $SO(3)$ is homeomorphic to $\mathbb{R}\mathrm{P}^3$. To see this, let $\varphi: D^3\to SO(3)$ send a nonzero vector $x$ to the rotation through angle $|x|\pi$ about the axis formed by the line through the origin in the direction of $x$. An orientation convention such as the 'right-hand rule' is needed to make this unambiguous. By continuity, $\varphi$ then sends 0 to the identity. Antipodal points of $S^2 = \partial D^3$ are sent to the same rotation through angle $\pi$, so $\varphi$ induces a map $\overline{\varphi}: \mathbb{R}\mathrm{P}^3\to SO(3)$, regarding $\mathbb{R}\mathrm{P}^3$ as $D^3$ with antipodal boundary points identified. The map $\overline{\varphi}$ is clearly injective since the axis of a nontrivial rotation is uniquely determined as its fixed point set, and $\overline{\varphi}$ is surjective since by easy linear algebra each nonidentity element

of $SO(3)$ is a rotation about some axis. It follows that $\overline{\varphi}$ is a homeomorphism $\mathbb{R}\mathrm{P}^3 \approx SO(3)$.

- $SO(4)$ is homeomorphic to $S^3 \times SO(3)$. Identifying $\mathbb{R}^4$ with the quaternions $\mathbb{H}$ and $S^3$ with the group of unit quaternions, the quaternion multiplication $v \mapsto vw$ for fixed $w \in S^3$ defines an isometry $\rho_w \in O(4)$ since $|vw| = |v||w| = |v|$ if $|w| = 1$. Points of $O(4)$ are 4-tuples $(v_1, \cdots, v_4)$ of orthonormal vectors $v_i \in \mathbb{H} = \mathbb{R}^4$, and we view $O(3)$ as the subspace with $v_1 = 1$. A homeomorphism $S^3 \times O(3) \to O(4)$ is defined by sending $(v, (1, v_2, v_3, v_4))$ to $(v, v_2 v, v_3 v, v_4 v) = \rho_v(1, v_2, v_3, v_4)$, with inverse $(v, v_2, v_3, v_4) \mapsto (v, (1, v_2 v^{-1}, v_3 v^{-1}, v_4 v^{-1})) = (v, \rho_{v^{-1}}(v, v_2, v_3, v_4))$. Restricting to identity components, we obtain a homeomorphism $S^3 \times SO(3) \approx SO(4)$. This is not a group isomorphism, however. It can be shown, though we will not digress to do so here, that the homomorphism $\psi : S^3 \times S^3 \to SO(4)$ sending a pair $(u, v)$ of unit quaternions to the isometry $w \mapsto uwv^{-1}$ of $\mathbb{H}$ is surjective with kernel $\mathbb{Z}_2 = \{\pm(1,1)\}$, and that $\psi$ is a covering space projection, representing $S^3 \times S^3$ as a 2-sheeted cover of $SO(4)$, the universal cover. Restricting $\psi$ to the diagonal $S^3 = \{(u,u)\} \subset S^3 \times S^3$ gives the universal cover $S^3 \to SO(3)$, so $SO(3)$ is isomorphic to the quotient group of $S^3$ by the normal subgroup $\{\pm 1\}$.

Using octonions one can construct in the same way a homeomorphism $SO(8) \approx S^7 \times SO(7)$. But in all other cases $SO(n)$ is only a 'twisted product' of $SO(n-1)$ and $S^{n-1}$; see Example 4.55 and the discussion following Corollary 4D.3.

### Cell Structure

Our first task is to construct a CW structure on $SO(n)$. This will come with a very nice cellular map $\rho : \mathbb{R}\mathrm{P}^{n-1} \times \mathbb{R}\mathrm{P}^{n-2} \times \cdots \times \mathbb{R}\mathrm{P}^1 \to SO(n)$. To simplify notation we will write $P^i$ for $\mathbb{R}\mathrm{P}^i$.

To each nonzero vector $v \in \mathbb{R}^n$ we can associate the reflection $r(v) \in O(n)$ across the hyperplane consisting of all vectors orthogonal to $v$. Since $r(v)$ is a reflection, it has determinant $-1$, so to get an element of $SO(n)$ we consider the composition $\rho(v) = r(v)r(e_1)$ where $e_1$ is the first standard basis vector $(1, 0, \cdots, 0)$. Since $\rho(v)$ depends only on the line spanned by $v$, $\rho$ defines a map $P^{n-1} \to SO(n)$. This map is injective since it is the composition of $v \mapsto r(v)$, which is obviously an injection of $P^{n-1}$ into $O(n) - SO(n)$, with the homeomorphism $O(n) - SO(n) \to SO(n)$ given by right-multiplication by $r(e_1)$. Since $\rho$ is injective and $P^{n-1}$ is compact Hausdorff, we may think of $\rho$ as embedding $P^{n-1}$ as a subspace of $SO(n)$.

More generally, for a sequence $I = (i_1, \cdots, i_m)$ with each $i_j < n$, we define a map $\rho : P^I = P^{i_1} \times \cdots \times P^{i_m} \to SO(n)$ by letting $\rho(v_1, \cdots, v_m)$ be the composition $\rho(v_1) \cdots \rho(v_m)$. If $\varphi^i : D^i \to P^i$ is the standard characteristic map for the $i$-cell of $P^i$, restricting to the 2-sheeted covering projection $\partial D^i \to P^{i-1}$, then the product $\varphi^I : D^I \to P^I$ of the appropriate $\varphi^{i_j}$'s is a characteristic map for the top-dimensional cell of $P^I$. We will be especially interested in the sequences $I = (i_1, \cdots, i_m)$ satisfying $n > i_1 > \cdots > i_m > 0$. These sequences will be called *admissible*, as will the sequence consisting of a single $0$.

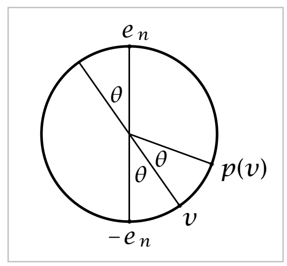
**Proposition 3D.1.** *The maps $\rho\varphi^I : D^I \to SO(n)$, for $I$ ranging over all admissible sequences, are the characteristic maps of a CW structure on $SO(n)$ for which the map $\rho : P^{n-1} \times P^{n-2} \times \cdots \times P^1 \to SO(n)$ is cellular.*

In particular, there is a single 0-cell $e^0 = \{\mathbb{1}\}$, so $SO(n)$ is path-connected. The other cells $e^I = e^{i_1} \cdots e^{i_m}$ are products, via the group operation in $SO(n)$, of the cells $e^i \subset P^{n-1} \subset SO(n)$.

**Proof**: According to Proposition A.2 in the Appendix, there are three things to show in order to obtain the CW structure:

(1) For each decreasing sequence $I$, $\rho\varphi^I$ is a homeomorphism from the interior of $D^I$ onto its image.
(2) The resulting image cells $e^I$ are all disjoint and cover $SO(n)$.
(3) For each $e^I$, $\rho\varphi^I(\partial D^I)$ is contained in a union of cells of lower dimension than $e^I$.

To begin the verification of these properties, define $p : SO(n) \to S^{n-1}$ by evaluation at the vector $e_n = (0, \cdots, 0, 1)$, $p(\alpha) = \alpha(e_n)$. Isometries in $P^{n-2} \subset P^{n-1} \subset SO(n)$ fix $e_n$, so $p(P^{n-2}) = \{e_n\}$. We claim that $p$ is a homeomorphism from $P^{n-1} - P^{n-2}$ onto $S^{n-1} - \{e_n\}$. This can be seen as follows. Thinking of a point in $P^{n-1}$ as a vector $v$, the map $p$ takes this to $\rho(v)(e_n) = r(v)r(e_1)(e_n)$, which equals $r(v)(e_n)$ since $e_n$ is in the hyperplane orthogonal to $e_1$. From the picture at the right it is then clear that $p$ simply stretches the lower half of each meridian circle in $S^{n-1}$ onto the whole meridian circle, doubling the angle up from the south pole, so $P^{n-1} - P^{n-2}$, represented by vectors whose last coordinate is negative, is taken homeomorphically onto $S^{n-1} - \{e_n\}$.




The next statement is that the map

$$h : (P^{n-1} \times SO(n-1), P^{n-2} \times SO(n-1)) \to (SO(n), SO(n-1)), \quad h(v, \alpha) = \rho(v)\alpha$$

is a homeomorphism from $(P^{n-1} - P^{n-2}) \times SO(n-1)$ onto $SO(n) - SO(n-1)$. Here we view $SO(n-1)$ as the subgroup of $SO(n)$ fixing the vector $e_n$. To construct an inverse to this homeomorphism, let $\beta \in SO(n) - SO(n-1)$ be given. Then $\beta(e_n) \neq e_n$ so by the preceding paragraph there is a unique $v_\beta \in P^{n-1} - P^{n-2}$ with $\rho(v_\beta)(e_n) = \beta(e_n)$, and $v_\beta$ depends continuously on $\beta$ since $\beta(e_n)$ does. The composition $\alpha_\beta = \rho(v_\beta)^{-1}\beta$ then fixes $e_n$, hence lies in $SO(n-1)$. Since $\rho(v_\beta)\alpha_\beta = \beta$, the map $\beta \mapsto (v_\beta, \alpha_\beta)$ is an inverse to $h$ on $SO(n) - SO(n-1)$.

Statements (1) and (2) can now be proved by induction on $n$. The map $\rho$ takes $P^{n-2}$ to $SO(n-1)$, so we may assume inductively that the maps $\rho\varphi^I$ for $I$ ranging over admissible sequences with first term $i_1 < n-1$ are the characteristic maps for a CW structure on $SO(n-1)$, with cells the corresponding products $e^I$. The admissible sequences $I$ with $i_1 = n-1$ then give disjoint cells $e^I$ covering $SO(n) - SO(n-1)$ by what was shown in the previous paragraph. So (1) and (2) hold for $SO(n)$.

To prove (3) it suffices to show there is an inclusion $P^iP^i \subset P^iP^{i-1}$ in $SO(n)$ since for an admissible sequence $I$, the map $\rho : P^I \to SO(n)$ takes the boundary of the top-dimensional cell of $P^I$ to the image of products $P^J$ with $J$ obtained from $I$ by decreasing one term $i_j$ by $1$, yielding a sequence which is admissible except perhaps for having two successive terms equal. As a preliminary to showing that $P^iP^i \subset P^iP^{i-1}$, observe that for $\alpha \in O(n)$ we have $r(\alpha(v)) = \alpha r(v)\alpha^{-1}$. Hence $\rho(v)\rho(w) = r(v)r(e_1)r(w)r(e_1) = r(v)r(w')$ where $w' = r(e_1)w$. Thus to show $P^iP^i \subset P^iP^{i-1}$ it suffices to find for each pair $v, w \in \mathbb{R}^{i+1}$ a pair $x \in \mathbb{R}^{i+1}$, $y \in \mathbb{R}^i$ with $r(v)r(w) = r(x)r(y)$.

Let $V \subset \mathbb{R}^{i+1}$ be a 2-dimensional subspace containing $v$ and $w$. Since $V \cap \mathbb{R}^i$ is at least 1-dimensional, we can choose a unit vector $y \in V \cap \mathbb{R}^i$. Let $\alpha \in O(i+1)$ take $V$ to $\mathbb{R}^2$ and $y$ to $e_1$. Then the conjugate $\alpha r(v)r(w)\alpha^{-1} = r(\alpha(v))r(\alpha(w))$ lies in $SO(2)$, hence has the form $\rho(z) = r(z)r(e_1)$ for some $z \in \mathbb{R}^2$ by statement (2) for $n = 2$. Therefore

$$r(v)r(w) = \alpha^{-1}r(z)r(e_1)\alpha = r(\alpha^{-1}(z))r(\alpha^{-1}(e_1)) = r(x)r(y)$$

for $x = \alpha^{-1}(z) \in \mathbb{R}^{i+1}$ and $y \in \mathbb{R}^i$.

It remains to show that the map $\rho : P^{n-1} \times P^{n-2} \times \cdots \times P^1 \to SO(n)$ is cellular. This follows from the inclusions $P^iP^i \subset P^iP^{i-1}$ derived above, together with another family of inclusions $P^iP^j \subset P^jP^i$ for $i < j$. To prove the latter we have the formulas

$$\begin{aligned}
\rho(v)\rho(w) &= r(v)r(w') && \text{where } w' = r(e_1)w, \text{ as earlier} \\
&= r(v)r(w')r(v)r(v) \\
&= r(r(v)w')r(v) && \text{from } r(\alpha(v)) = \alpha r(v)\alpha^{-1} \\
&= r(r(v)r(e_1)w)r(v) = r(\rho(v)w)r(v) \\
&= \rho(\rho(v)w)\rho(v') && \text{where } v' = r(e_1)v, \text{ hence } v = r(e_1)v'
\end{aligned}$$

In particular, taking $v \in \mathbb{R}^{i+1}$ and $w \in \mathbb{R}^{j+1}$ with $i < j$, we have $\rho(v)w \in \mathbb{R}^{j+1}$, and the product $\rho(v)\rho(w) \in P^iP^j$ equals the product $\rho(\rho(v)w)\rho(v') \in P^jP^i$. $\square$

### Mod 2 Homology and Cohomology

Each cell of $SO(n)$ is the homeomorphic image of a cell in $P^{n-1} \times P^{n-2} \times \cdots \times P^1$, so the cellular chain map induced by $\rho : P^{n-1} \times P^{n-2} \times \cdots \times P^1 \to SO(n)$ is surjective. It follows that with $\mathbb{Z}_2$ coefficients the cellular boundary maps for $SO(n)$ are all trivial since this is true in $P^i$ and hence in $P^{n-1} \times P^{n-2} \times \cdots \times P^1$ by Proposition 3B.1. Thus $H_*(SO(n);\mathbb{Z}_2)$ has a $\mathbb{Z}_2$ summand for each cell of $SO(n)$. One can rephrase this as saying that there are isomorphisms $H_i(SO(n);\mathbb{Z}_2) \approx H_i(S^{n-1}\times S^{n-2}\times\cdots\times S^1;\mathbb{Z}_2)$ for all $i$ since this product of spheres also has cells in one-to-one correspondence with admissible sequences. The full structure of the $\mathbb{Z}_2$ homology and cohomology rings is given by:

**Theorem 3D.2.** **(a)** *$H^*(SO(n);\mathbb{Z}_2) \approx \bigotimes_{i\text{ odd}} \mathbb{Z}_2[\beta_i]/(\beta_i^{p_i})$ where $|\beta_i| = i$ and $p_i$ is the smallest power of $2$ such that $|\beta_i^{p_i}| \ge n$.*
**(b)** *The Pontryagin ring $H_*(SO(n);\mathbb{Z}_2)$ is the exterior algebra $\Lambda_{\mathbb{Z}_2}[e^1,\cdots,e^{n-1}]$.*

Here $e^i$ denotes the cellular homology class of the cell $e^i \subset P^{n-1} \subset SO(n)$, and $\beta_i$ is the dual class to $e^i$, represented by the cellular cochain assigning the value 1 to the cell $e^i$ and 0 to all other $i$-cells.

**Proof**: As we noted above, $\rho$ induces a surjection on cellular chains. Since the cellular boundary maps with $\mathbb{Z}_2$ coefficients are trivial for both $P^{n-1}\times\cdots\times P^1$ and $SO(n)$, it follows that $\rho_*$ is surjective on $H_*(-;\mathbb{Z}_2)$ and $\rho^*$ is injective on $H^*(-;\mathbb{Z}_2)$. We know that $H^*(P^{n-1}\times\cdots\times P^1;\mathbb{Z}_2)$ is the polynomial ring $\mathbb{Z}_2[\alpha_1,\cdots,\alpha_{n-1}]$ truncated by the relations $\alpha_i^{i+1} = 0$. For $\beta_i \in H^i(SO(n);\mathbb{Z}_2)$ the dual class to $e^i$, we have $\rho^*(\beta_i) = \sum_j \alpha_j^i$, the class assigning $1$ to each $i$-cell in a factor $P^j$ of $P^{n-1}\times\cdots\times P^1$ and $0$ to all other $i$-cells, which are products of lower-dimensional cells and hence map to cells in $SO(n)$ disjoint from $e^i$.

First we will show that the monomials $\beta_I = \beta_{i_1}\cdots\beta_{i_m}$ corresponding to admissible sequences $I$ are linearly independent in $H^*(SO(n);\mathbb{Z}_2)$, hence are a vector space basis. Since $\rho^*$ is injective, we may identify each $\beta_i$ with its image $\sum_j \alpha_j^i$ in the truncated polynomial ring $\mathbb{Z}_2[\alpha_1,\cdots,\alpha_{n-1}]/(\alpha_1^2,\cdots,\alpha_{n-1}^n)$. Suppose we have a linear relation $\sum_I b_I\beta_I = 0$ with $b_I \in \mathbb{Z}_2$ and $I$ ranging over the admissible sequences. Since each $\beta_I$ is a product of distinct $\beta_i$'s, we can write the relation in the form $x\beta_1 + y = 0$ where neither $x$ nor $y$ has $\beta_1$ as a factor. Since $\alpha_1$ occurs only in the term $\beta_1$ of $x\beta_1 + y$, where it has exponent $1$, we have $x\beta_1 + y = x\alpha_1 + z$ where neither $x$ nor $z$ involves $\alpha_1$. The relation $x\alpha_1 + z = 0$ in $\mathbb{Z}_2[\alpha_1,\cdots,\alpha_{n-1}]/(\alpha_1^2,\cdots,\alpha_{n-1}^n)$ then implies $x = 0$. Thus we may assume the original relation does not involve $\beta_1$. Now we repeat the argument for $\beta_2$. Write the relation in the form $x\beta_2 + y = 0$ where neither $x$ nor $y$ involves $\beta_2$ or $\beta_1$. The variable $\alpha_2$ now occurs only in the term $\beta_2$ of $x\beta_2 + y$, where it has exponent $2$, so we have $x\beta_2 + y = x\alpha_2^2 + z$ where $x$ and $z$ do not involve $\alpha_1$ or $\alpha_2$. Then $x\alpha_2^2 + z = 0$ implies $x = 0$ and we have a relation involving neither $\beta_1$ nor $\beta_2$. Continuing inductively, we eventually deduce that all coefficients $b_I$ in the original relation $\sum_I b_I\beta_I = 0$ must be zero.

Observe now that $\beta_i^2 = \beta_{2i}$ if $2i < n$ and $\beta_i^2 = 0$ if $2i \ge n$, since $\big(\sum_j \alpha_j^i\big)^2 = \sum_j \alpha_j^{2i}$. The quotient $Q$ of the algebra $\mathbb{Z}_2[\beta_1,\beta_2,\cdots]$ by the relations $\beta_i^2 = \beta_{2i}$ and $\beta_j = 0$ for $j \ge n$ then maps onto $H^*(SO(n);\mathbb{Z}_2)$. This map $Q \to H^*(SO(n);\mathbb{Z}_2)$ is also injective since the relations defining $Q$ allow every element of $Q$ to be represented as a linear combination of admissible monomials $\beta_I$, and the admissible monomials are linearly independent in $H^*(SO(n);\mathbb{Z}_2)$. The algebra $Q$ can also be described as the tensor product in statement (a) of the theorem since the relations $\beta_i^2 = \beta_{2i}$ allow admissible monomials to be written uniquely as monomials in powers of the $\beta_i$'s with $i$ odd, and the relation $\beta_j = 0$ for $j \ge n$ becomes $\beta_{ip_i} = \beta_i^{p_i} = 0$ where $j = ip_i$ with $i$ odd and $p_i$ a power of $2$. For a given $i$, this relation holds iff $ip_i \ge n$, or in other words, iff $|\beta_i^{p_i}| \ge n$. This finishes the proof of (a).

For part (b), note first that the group multiplication $SO(n)\times SO(n)\to SO(n)$ is cellular in view of the inclusions $P^iP^i \subset P^iP^{i-1}$ and $P^iP^j \subset P^jP^i$ for $i < j$. So we can compute Pontryagin products at the cellular level. We know that there is at least an additive isomorphism $H_*(SO(n);\mathbb{Z}_2) \approx \Lambda_{\mathbb{Z}_2}[e^1,\cdots,e^{n-1}]$ since the products $e^I = e^{i_1}\cdots e^{i_m}$ with $I$ admissible form a basis for $H_*(SO(n);\mathbb{Z}_2)$. The inclusion $P^iP^i \subset P^iP^{i-1}$ then implies that the Pontryagin product $(e^i)^2$ is $0$. It remains only to see the commutativity relation $e^ie^j = e^je^i$. The inclusion $P^iP^j \subset P^jP^i$ for $i<j$ was obtained from the formula $\rho(v)\rho(w) = \rho(\rho(v)w)\rho(v')$ for $v \in \mathbb{R}^{i+1}$, $w \in \mathbb{R}^{j+1}$, and $v' = r(e_1)v$. The map $f\colon P^i\times P^j \to P^j\times P^i$, $f(v,w) = (\rho(v)w, v')$, is a homeomorphism since it is the composition of homeomorphisms $(v,w)\mapsto (v,\rho(v)w)\mapsto (v',\rho(v)w)\mapsto (\rho(v)w,v')$. The first of these maps takes $e^i\times e^j$ homeomorphically onto itself since $\rho(v)(e^j) = e^j$ if $i<j$. Obviously the second map also takes $e^i\times e^j$ homeomorphically onto itself, while the third map simply transposes the two factors. Thus $f$ restricts to a homeomorphism from $e^i\times e^j$ onto $e^j\times e^i$, and therefore $e^ie^j = e^je^i$ in $H_*(SO(n);\mathbb{Z}_2)$. $\square$

The cup product and Pontryagin product structures in this theorem may seem at first glance to be unrelated, but in fact the relationship is fairly direct. As we saw in the previous section, the dual of a polynomial algebra $\mathbb{Z}_2[x]$ is a divided polynomial algebra $\Gamma_{\mathbb{Z}_2}[\alpha]$, and with $\mathbb{Z}_2$ coefficients the latter is an exterior algebra $\Lambda_{\mathbb{Z}_2}[\alpha_0,\alpha_1,\cdots]$ where $|\alpha_i| = 2^i|x|$. If we truncate the polynomial algebra by a relation $x^{2^n} = 0$, then this just eliminates the generators $\alpha_i$ for $i \ge n$. In view of this, if it were the case that the generators $\beta_i$ for the algebra $H^*(SO(n);\mathbb{Z}_2)$ happened to be primitive, then $H^*(SO(n);\mathbb{Z}_2)$ would be isomorphic as a Hopf algebra to the tensor product of the single-generator Hopf algebras $\mathbb{Z}_2[\beta_i]/(\beta_i^{p_i})$, $i = 1,3,\cdots$, hence the dual algebra $H_*(SO(n);\mathbb{Z}_2)$ would be the tensor product of the corresponding truncated divided polynomial algebras, in other words an exterior algebra as just explained. This is in fact the structure of $H_*(SO(n);\mathbb{Z}_2)$, so since the Pontryagin product in $H_*(SO(n);\mathbb{Z}_2)$ determines the coproduct in $H^*(SO(n);\mathbb{Z}_2)$ uniquely, it follows that the $\beta_i$'s must indeed be primitive.

It is not difficult to give a direct argument that each $\beta_i$ is primitive. The coproduct $\Delta\colon H^*(SO(n);\mathbb{Z}_2)\to H^*(SO(n);\mathbb{Z}_2)\otimes H^*(SO(n);\mathbb{Z}_2)$ is induced by the group multiplication $\mu\colon SO(n)\times SO(n)\to SO(n)$. We need to show that the value of $\Delta(\beta_i)$ on $e^I\otimes e^J$, which we denote $\langle\Delta(\beta_i), e^I\otimes e^J\rangle$, is the same as the value $\langle \beta_i\otimes 1 + 1\otimes\beta_i, e^I\otimes e^J\rangle$

for all cells $e^I$ and $e^J$ whose dimensions add up to $i$. Since $\Delta = \mu^*$, we have $\langle \Delta(\beta_i), e^I \otimes e^J \rangle = \langle \beta_i, \mu_*(e^I \otimes e^J) \rangle$. Because $\mu$ is the multiplication map, $\mu(e^I \times e^J)$ is contained in $P^I P^J$, and if we use the relations $P^j P^j \subset P^j P^{j-1}$ and $P^j P^k \subset P^k P^j$ for $j < k$ to rearrange the factors $P^j$ of $P^I P^J$ so that their dimensions are in decreasing order, then the only way we will end up with a term $P^i$ is if we start with $P^I P^J$ equal to $P^i P^0$ or $P^0 P^i$. Thus $\langle \beta_i, \mu_*(e^I \otimes e^J) \rangle = 0$ unless $e^I \otimes e^J$ equals $e^i \otimes e^0$ or $e^0 \otimes e^i$. Hence $\Delta(\beta_i)$ contains no other terms besides $\beta_i \otimes 1 + 1 \otimes \beta_i$, and $\beta_i$ is primitive.

### Integer Homology and Cohomology

With $\mathbb{Z}$ coefficients the homology and cohomology of $SO(n)$ turns out to be a good bit more complicated than with $\mathbb{Z}_2$ coefficients. One can see a little of this complexity already for small values of $n$, where the homeomorphisms $SO(3) \approx \mathbb{R}\mathrm{P}^3$ and $SO(4) \approx S^3 \times \mathbb{R}\mathrm{P}^3$ would allow one to compute the additive structure as a direct sum of a certain number of $\mathbb{Z}$'s and $\mathbb{Z}_2$'s. For larger values of $n$ the additive structure is qualitatively the same:

**Proposition 3D.3.** *$H_*(SO(n);\mathbb{Z})$ is a direct sum of $\mathbb{Z}$'s and $\mathbb{Z}_2$'s.*

**Proof**: We compute the cellular chain complex of $SO(n)$, showing that it splits as a tensor product of simpler complexes. For a cell $e^i \subset P^{n-1} \subset SO(n)$ the cellular boundary $de^i$ is $2e^{i-1}$ for even $i > 0$ and $0$ for odd $i$. To compute the cellular boundary of a cell $e^{i_1} \cdots e^{i_m}$ we can pull it back to a cell $e^{i_1} \times \cdots \times e^{i_m}$ of $P^{n-1} \times \cdots \times P^1$ whose cellular boundary, by Proposition 3B.1, is $\sum_j (-1)^{\sigma_j} e^{i_1} \times \cdots \times de^{i_j} \times \cdots \times e^{i_m}$ where $\sigma_j = i_1 + \cdots + i_{j-1}$. Hence $d(e^{i_1} \cdots e^{i_m}) = \sum_j (-1)^{\sigma_j} e^{i_1} \cdots de^{i_j} \cdots e^{i_m}$, where it is understood that $e^{i_1} \cdots de^{i_j} \cdots e^{i_m}$ is zero if $i_j = i_{j+1} + 1$ since $P^{i_j - 1} P^{i_j - 1} \subset P^{i_j - 1} P^{i_j - 2}$, in a lower-dimensional skeleton.

To split the cellular chain complex $C_*(SO(n))$ as a tensor product of smaller chain complexes, let $C^{2i}$ be the subcomplex of $C_*(SO(n))$ with basis the cells $e^0$, $e^{2i}$, $e^{2i-1}$, and $e^{2i}e^{2i-1}$. This is a subcomplex since $de^{2i-1} = 0$, $de^{2i} = 2e^{2i-1}$, and, in $P^{2i} \times P^{2i-1}$, $d(e^{2i} \times e^{2i-1}) = de^{2i} \times e^{2i-1} + e^{2i} \times de^{2i-1} = 2e^{2i-1} \times e^{2i-1}$, hence $d(e^{2i}e^{2i-1}) = 0$ since $P^{2i-1}P^{2i-1} \subset P^{2i-1}P^{2i-2}$. The claim is that there are chain complex isomorphisms

$$C_*(SO(2k+1)) \approx C^2 \otimes C^4 \otimes \cdots \otimes C^{2k}$$
$$C_*(SO(2k+2)) \approx C^2 \otimes C^4 \otimes \cdots \otimes C^{2k} \otimes C^{2k+1}$$

where $C^{2k+1}$ has basis $e^0$ and $e^{2k+1}$. Certainly these isomorphisms hold for the chain groups themselves, so it is only a matter of checking that the boundary maps agree. For the case of $C_*(SO(2k+1))$ this can be seen by induction on $k$, as the reader can easily verify. Then the case of $C_*(SO(2k+2))$ reduces to the first case by a similar argument.

Since $H_*(C^{2i})$ consists of $\mathbb{Z}$'s in dimensions $0$ and $4i-1$ and a $\mathbb{Z}_2$ in dimension $2i-1$, while $H_*(C^{2k+1})$ consists of $\mathbb{Z}$'s in dimensions $0$ and $2k+1$, we conclude from the algebraic Künneth formula that $H_*(SO(n);\mathbb{Z})$ is a direct sum of $\mathbb{Z}$'s and $\mathbb{Z}_2$'s. □

Note that the calculation shows that $SO(2k)$ and $SO(2k-1)\times S^{2k-1}$ have isomorphic homology groups in all dimensions.

In view of the preceding proposition, one can get rather complete information about $H_*(SO(n);\mathbb{Z})$ by considering the natural maps to $H_*(SO(n);\mathbb{Z}_2)$ and to the quotient of $H_*(SO(n);\mathbb{Z})$ by its torsion subgroup. Let us denote this quotient by $H_*^{free}(SO(n);\mathbb{Z})$. The same strategy applies equally well to cohomology, and the universal coefficient theorem gives an isomorphism $H^*_{free}(SO(n);\mathbb{Z}) \approx H_*^{free}(SO(n);\mathbb{Z})$.

The proof of the proposition shows that the additive structure of $H_*^{free}(SO(n);\mathbb{Z})$ is fairly simple:

$$H_*^{free}(SO(2k+1);\mathbb{Z}) \approx H_*(S^3\times S^7\times\cdots\times S^{4k-1})$$
$$H_*^{free}(SO(2k+2);\mathbb{Z}) \approx H_*(S^3\times S^7\times\cdots\times S^{4k-1}\times S^{2k+1})$$

The multiplicative structure is also as simple as it could be:

**Proposition 3D.4.** *The Pontryagin ring $H_*^{free}(SO(n);\mathbb{Z})$ is an exterior algebra,*

$$H_*^{free}(SO(2k+1);\mathbb{Z}) \approx \Lambda_{\mathbb{Z}}[a_3,a_7,\cdots,a_{4k-1}] \quad \text{where } |a_i| = i$$
$$H_*^{free}(SO(2k+2);\mathbb{Z}) \approx \Lambda_{\mathbb{Z}}[a_3,a_7,\cdots,a_{4k-1},a'_{2k+1}]$$

*The generators $a_i$ and $a'_{2k+1}$ are primitive, so the dual Hopf algebra $H^*_{free}(SO(n);\mathbb{Z})$ is an exterior algebra on the dual generators $\alpha_i$ and $\alpha'_{2k+1}$.*

**Proof**: As in the case of $\mathbb{Z}_2$ coefficients we can work at the level of cellular chains since the multiplication in $SO(n)$ is cellular. Consider first the case $n = 2k+1$. Let $E^i$ be the cycle $e^{2i}e^{2i-1}$ generating a $\mathbb{Z}$ summand of $H_*(SO(n);\mathbb{Z})$. By what we have shown above, the products $E^{i_1}\cdots E^{i_m}$ with $i_1 > \cdots > i_m$ form an additive basis for $H_*^{free}(SO(n);\mathbb{Z})$, so we need only verify that the multiplication is as in an exterior algebra on the classes $E^i$. The map $f$ in the proof of Theorem 3D.2 gives a homeomorphism $e^i\times e^j \approx e^j\times e^i$ if $i<j$, and this homeomorphism has local degree $(-1)^{ij+1}$ since it is the composition $(v,w)\mapsto (v,\rho(v)w)\mapsto (v',\rho(v)w)\mapsto (\rho(v)w,v')$ of homeomorphisms with local degrees $+1,-1$, and $(-1)^{ij}$. Applying this four times to commute $E^iE^j = e^{2i}e^{2i-1}e^{2j}e^{2j-1}$ to $E^jE^i = e^{2j}e^{2j-1}e^{2i}e^{2i-1}$, three of the four applications give a sign of $-1$ and the fourth gives a $+1$, so we conclude that $E^iE^j = -E^jE^i$ if $i<j$. When $i=j$ we have $(E^i)^2 = 0$ since $e^{2i}e^{2i-1}e^{2i}e^{2i-1} = e^{2i}e^{2i}e^{2i-1}e^{2i-1}$, which lies in a lower-dimensional skeleton because of the relation $P^{2i}P^{2i}\subset P^{2i}P^{2i-1}$.

Thus we have shown that $H_*(SO(2k+1);\mathbb{Z})$ contains $\Lambda_{\mathbb{Z}}[E^1,\cdots,E^k]$ as a subalgebra. The same reasoning shows that $H_*(SO(2k+2);\mathbb{Z})$ contains the subalgebra $\Lambda_{\mathbb{Z}}[E^1,\cdots,E^k,e^{2k+1}]$. These exterior subalgebras account for all the nontorsion in $H_*(SO(n);\mathbb{Z})$, so the product structure in $H_*^{free}(SO(n);\mathbb{Z})$ is as stated.

Now we show that the generators $E^i$ and $e^{2k+1}$ are primitive in $H_*^{free}(SO(n);\mathbb{Z})$. Looking at the formula for the boundary maps in the cellular chain complex of $SO(n)$, we see that this chain complex is the direct sum of the subcomplexes $C(m)$ with basis the $m$-fold products $e^{i_1}\cdots e^{i_m}$ with $i_1 > \cdots > i_m > 0$. We allow $m = 0$ here, with $C(0)$ having basis the 0-cell of $SO(n)$. The direct sum $C(0)\oplus\cdots\oplus C(m)$ is the cellular chain complex of the subcomplex of $SO(n)$ consisting of cells that are products of $m$ or fewer cells $e^i$. In particular, taking $m = 2$ we have a subcomplex $X \subset SO(n)$ whose homology, mod torsion, consists of the $\mathbb{Z}$ in dimension zero and the $\mathbb{Z}$'s generated by the cells $E^i$, together with the cell $e^{2k+1}$ when $n = 2k+2$. The inclusion $X \hookrightarrow SO(n)$ induces a commutative diagram

$$\begin{array}{ccc} H_*^{free}(X;\mathbb{Z}) & \xrightarrow{\ \Delta\ } & H_*^{free}(X;\mathbb{Z}) \otimes H_*^{free}(X;\mathbb{Z}) \\ \downarrow & & \downarrow \\ H_*^{free}(SO(n);\mathbb{Z}) & \xrightarrow{\ \Delta\ } & H_*^{free}(SO(n);\mathbb{Z}) \otimes H_*^{free}(SO(n);\mathbb{Z}) \end{array}$$

where the lower $\Delta$ is the coproduct in $H_*^{free}(SO(n);\mathbb{Z})$ and the upper $\Delta$ is its analog for $X$, coming from the diagonal map $X \to X\times X$ and the Künneth formula. The classes $E^i$ in the lower left group pull back to elements we label $\widetilde{E}^i$ in the upper left group. Since these have odd dimension and $H_*^{free}(X;\mathbb{Z})$ vanishes in even positive dimensions, the images $\Delta(\widetilde{E}^i)$ can have no components $a\otimes b$ with both $a$ and $b$ positive-dimensional. The same is therefore true for $\Delta(E^i)$ by commutativity of the diagram, so the classes $E^i$ are primitive. This argument also works for $e^{2k+1}$ when $n = 2k+2$.

Since the exterior algebra generators of $H_*^{free}(SO(n);\mathbb{Z})$ are primitive, this algebra splits as a Hopf algebra into a tensor product of single-generator exterior algebras $\Lambda_{\mathbb{Z}}[a_i]$ (and $\Lambda_{\mathbb{Z}}[a'_{2k+1}]$). The dual Hopf algebra $H^*_{free}(SO(n);\mathbb{Z})$ therefore splits as the tensor product of the dual exterior algebras $\Lambda_{\mathbb{Z}}[\alpha_i]$ (and $\Lambda_{\mathbb{Z}}[\alpha'_{2k+1}]$), hence $H^*_{free}(SO(n);\mathbb{Z})$ is also an exterior algebra. $\square$

The exact ring structure of $H^*(SO(n);\mathbb{Z})$ can be deduced from these results via Bockstein homomorphisms, as we show in Example 3E.7, though the process is somewhat laborious and the answer not very neat.

## Stiefel Manifolds

Consider the **Stiefel manifold** $V_{n,k}$, whose points are the *orthonormal k-frames* in $\mathbb{R}^n$, that is, orthonormal $k$-tuples of vectors. Thus $V_{n,k}$ is a subset of the product of $k$ copies of $S^{n-1}$, and it is given the subspace topology. As special cases, $V_{n,n} = O(n)$ and $V_{n,1} = S^{n-1}$. Also, $V_{n,2}$ can be identified with the space of unit tangent vectors to $S^{n-1}$ since a vector $v$ at the point $x \in S^{n-1}$ is tangent to $S^{n-1}$ iff it is orthogonal to $x$. We can also identify $V_{n,n-1}$ with $SO(n)$ since there is a unique way of extending an orthonormal $(n-1)$-frame to a positively oriented orthonormal $n$-frame.

There is a natural projection $p: O(n) \to V_{n,k}$ sending $\alpha \in O(n)$ to the $k$-frame consisting of the last $k$ columns of $\alpha$, which are the images under $\alpha$ of the last $k$ standard basis vectors in $\mathbb{R}^n$. This projection is onto, and the preimages of points are precisely the cosets $\alpha O(n-k)$, where we embed $O(n-k)$ in $O(n)$ as the orthogonal transformations of the first $n-k$ coordinates of $\mathbb{R}^n$. Thus $V_{n,k}$ can be viewed as the space $O(n)/O(n-k)$ of such cosets, with the quotient topology from $O(n)$. This is the same as the previously defined topology on $V_{n,k}$ since the projection $O(n) \to V_{n,k}$ is a surjection of compact Hausdorff spaces.

When $k < n$ the projection $p: SO(n) \to V_{n,k}$ is surjective, and $V_{n,k}$ can also be viewed as the coset space $SO(n)/SO(n-k)$. We can use this to induce a CW structure on $V_{n,k}$ from the CW structure on $SO(n)$. The cells are the sets of cosets of the form $e^I SO(n-k) = e^{i_1} \cdots e^{i_m} SO(n-k)$ for $n > i_1 > \cdots > i_m \geq n-k$, together with the coset $SO(n-k)$ itself as a 0-cell of $V_{n,k}$. These sets of cosets are unions of cells of $SO(n)$ since $SO(n-k)$ consists of the cells $e^J = e^{j_1} \cdots e^{j_\ell}$ with $n-k > j_1 > \cdots > j_\ell$. This implies that $V_{n,k}$ is the disjoint union of its cells, and the boundary of each cell is contained in cells of lower dimension, so we do have a CW structure.

Since the projection $SO(n) \to V_{n,k}$ is a cellular map, the structure of the cellular chain complex of $V_{n,k}$ can easily be deduced from that of $SO(n)$. For example, the cellular chain complex of $V_{2k+1,2}$ is just the complex $C^{2k}$ defined earlier, while for $V_{2k,2}$ the cellular boundary maps are all trivial. Hence the nonzero homology groups of $V_{n,2}$ are

$$H_i(V_{2k+1,2};\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } i = 0,\ 4k-1 \\ \mathbb{Z}_2 & \text{for } i = 2k-1 \end{cases}$$

$$H_i(V_{2k,2};\mathbb{Z}) = \mathbb{Z} \quad \text{for } i = 0,\ 2k-2,\ 2k-1,\ 4k-3$$

Thus $SO(n)$ has the same homology and cohomology groups as the product space $V_{3,2} \times V_{5,2} \times \cdots \times V_{2k+1,2}$ when $n = 2k+1$, or as $V_{3,2} \times V_{5,2} \times \cdots \times V_{2k+1,2} \times S^{2k+1}$ when $n = 2k+2$. However, our calculations show that $SO(n)$ is distinguished from these products by its cup product structure with $\mathbb{Z}_2$ coefficients, at least when $n \geq 5$, since $\beta_1^4$ is nonzero in $H^4(SO(n);\mathbb{Z}_2)$ if $n \geq 5$, while for the product spaces the nontrivial element of $H^1(-;\mathbb{Z}_2)$ must lie in the factor $V_{3,2}$, and $H^4(V_{3,2};\mathbb{Z}_2) = 0$. When $n = 4$ we have $SO(4)$ homeomorphic to $SO(3) \times S^3 = V_{3,2} \times S^3$ as we noted at the beginning of this section. Also $SO(3) = V_{3,2}$ and $SO(2) = S^1$.

## Exercises

**1.** Show that a topological group with a finite-dimensional CW structure is an orientable manifold. [Consider the homeomorphisms $x \mapsto gx$ or $x \mapsto xg$ for fixed $g$ and varying $x$ in the group.]

**2.** Using the CW structure on $SO(n)$, show that $\pi_1 SO(n) \approx \mathbb{Z}_2$ for $n \geq 3$. Find a loop representing a generator, and describe how twice this loop is nullhomotopic.

**3.** Compute the Pontryagin ring structure in $H_*(SO(5);\mathbb{Z})$.