# Resolving powers; Chebyshev, ~~Whitehead~~

We saw in the exercises that for $(0 \leqslant \alpha \leqslant 1)$,

$$a^{\alpha} b^{1-\alpha} \leqslant \alpha a + (1-\alpha) b$$

with equality only if $a=b$, $\alpha=0$ or $\alpha=1$.
(Recall that the lhs is $\mathcal{G}((a,b))$ and the rhs is $\mathcal{U}((a,b))$ with weights $\alpha$ and $1-\alpha$.)
We can rewrite this as

$$a^{\alpha} \leqslant (\alpha a + (1-\alpha) b) b^{\alpha-1}$$

$$\begin{aligned} a^{\alpha} - b^{\alpha} &\leqslant (\alpha a + (1-\alpha) b) b^{\alpha-1} - b^{\alpha} \\ &\leqslant (\alpha a + (1-\alpha) b - b) b^{\alpha-1} \\ &\leqslant (\alpha a - \alpha b) b^{\alpha-1} \\ &\leqslant \alpha (a-b) b^{\alpha-1} \end{aligned}$$

with equality only if $\alpha=0$, $\alpha=1$ or $a=b$.

Symmetrically,

$$b^\alpha - a^\alpha \leq \alpha(b-a)a^{\alpha-1}$$

so we have (for $0<\alpha<1$)

$$\alpha(a-b)b^{\alpha-1} \geq a^\alpha - b^\alpha \geq \alpha(a-b)a^{\alpha-1}$$

If we have two numbers $\{a,b\}$ and let $M = \max\{a,b\}$, $m = \min\{a,b\}$ it's helpful to write the above as

$$\alpha m^{\alpha-1} \geq \frac{M^\alpha - m^\alpha}{M-m} \geq \alpha M^{\alpha-1} \qquad (0<\alpha<1)$$

for $\alpha<0$ or $\alpha>1$, the sense reverses and (exercise)

$$\alpha M^{\alpha-1} \geq \frac{M^\alpha - m^\alpha}{M-m} \geq \alpha m^{\alpha-1}$$

N.b. For $\alpha=2$, $\frac{M^2-m^2}{M-m} = M+m$, which is clearly between $2M \geq M+m \geq 2m$, with equality $\Leftrightarrow M=m$.

We know from ~~Hölder~~ Minkowski's inequality that $M_r(\vec{a}) + M_r(\vec{b})$ is either $\geq$ or $\leq$ $M_r(\vec{a}+\vec{b})$ depending on whether $r > 1$ or $r < 1$. What about $M_r(\vec{a})M_r(\vec{b})$ and $M_r(\vec{a}\vec{b})$? $\cancel{\leq}$

Definition. If there is a (common) permutation of indices which makes $a_i$ and $b_i$ into non-decreasing sequences, we say $\vec{a}, \vec{b}$ are <u>similarly ordered</u>.

If there is a common permutation of indices which makes $a_i$ into a non-decreasing sequence and $b_i$ into a non-increasing sequence, we say $\vec{a}, \vec{b}$ are <u>oppositely ordered</u>.

**Chebyshev's Inequality.** If $r>0$ and $\vec{a},\vec{b}$ are similarly ordered,

$$M_r(\vec{a})\,M_r(\vec{b}) \leq M_r(\vec{a}\vec{b})$$

with equality only if (all $a_i$ are equal) or (all $b_i$ are equal). If $\vec{a},\vec{b}$ are oppositely ordered

$$M_r(\vec{a})\,M_r(\vec{b}) \geq M_r(\vec{a}\vec{b})$$

*Proof.* We start with the case $r=1$. <u>Without</u> assuming $\sum p_i = 1$,

$$\left(M_1(\vec{a}\vec{b}) - M_1(\vec{a})M_1(b)\right)\left(\sum p_i\right)^2$$

$$= \sum_i p_i \sum_j p_j a_j b_j - \sum_i p_i a_i \sum_j p_j b_j$$

$$= \sum_{i,j} p_i p_j a_j b_j - p_i p_j a_i b_j$$

Now this is equal to

$$\sum_{i,j} p_i p_j a_i b_i - p_i p_j a_j b_i$$

by swapping indices i and j. So we can write the sum as

$$= \frac{1}{2} \sum_{i,j} p_i p_j (a_j b_j - a_i b_j + a_i b_i - a_j b_i)$$

$$= \frac{1}{2} \sum_{i,j} p_i p_j (a_j - a_i)(b_j - b_i)$$

Now if ~~the~~ $\vec{a}, \vec{b}$ are similarly ordered, we can assume wlog that we have permuted indices so $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are nondecreasing.

In this case, $a_j - a_i \geq 0$ if $j > i$

$a_j - a_i \leq 0$ if $j < i$

**and** the same holds for $b_j - b_i$.

Either way, $(a_j - a_i)(b_j - b_i) \geq 0$, so

$$\frac{1}{2}\sum p_i p_j (a_j - a_i)(b_j - b_i) \geq 0$$

We have equality only if all these terms are 0, including

$$p_1 p_n (a_n - a_1)(b_n - b_1)$$

which can only be zero if $a_1 = a_n$ (and hence all $a_i$ are equal) or $b_1 = b_n$ (and hence all $a_j$ are equal).

Of course, if $\vec{a}, \vec{b}$ are <u>oppositely</u> ordered, the signs of $(a_j - a_i), (b_j - b_i)$ always <u>disagree</u> and the sum is $\leq 0$. The case of equality is the same.

Recalling that

$$M_r(\vec{a})^r = M_1(\vec{a}^r)$$

we get

$$M_r(\vec{a})^r M_r(\vec{b})^r \leq M_r(\vec{a}\vec{b})^r \quad \text{(sim. ordered)}$$

$$M_r(\vec{a})^r M_r(\vec{b})^r \geq M_r(\vec{a}\vec{b})^r \quad \text{(opp. ordered)}$$

immediately. Since $r>0$, taking the $r$-th root preserves the sense of the inequalities. $\square$

Note. Since $\vec{a}$ is similarly ordered with itself, Chebyshev implies

$$M_r(\vec{a})^m \leq M_r(\vec{a}^m)$$

or

$$M_r(\vec{a}) \leq M_r(\vec{a}^m)^{1/m} = (U(\vec{a}^{mr}))^{1/mr} = M_{mr}(\vec{a})$$

which is a sobcase of the general $r<s \Rightarrow M_r(\vec{a}) \leq M_s(\vec{a})$ theorem.

Pause to reflect: We have shown that $M_r(\vec{a}\vec{b})$ is <u>not</u> generally comparable to $M_r(\vec{a})M_r(\vec{b})$. Could it be comparable to $M_s(\vec{a})M_t(\vec{b})$ for some $s$ and $t$?

Theorem. $M_{r_0}(\vec{a}_1 \cdots \vec{a}_m)$ is generally comparable to $M_{r_1}(\vec{a}_1) \cdots M_{r_m}(\vec{a}_m)$ (for positive $r_0, \ldots, r_m$) if and only if

$$\frac{1}{r_0} \geq \frac{1}{r_1} + \cdots + \frac{1}{r_m}$$

in which case

$$M_{r_0}(\vec{a}_1 \cdots \vec{a}_m) \leq M_{r_1}(\vec{a}_1) \cdots M_{r_m}(\vec{a}_m).$$

Sanity check: $\frac{1}{r} < \frac{1}{r} + \frac{1}{r}$, so this does not contradict Chebyshev.

Proof. We recall Hölder's inequality

$$M_r(\vec{a}_1 \cdots \vec{a}_m) \leq M_{r/p_1}(\vec{a}_1) \cdots M_{r/p_m}(\vec{a}_m)$$

when $p_1 + \dots + p_m = 1$. In this case

$$\frac{1}{r} = \frac{1}{r/p_1} + \dots + \frac{1}{r/p_m} = \frac{p_1}{r} + \dots + \frac{p_m}{r} = \frac{1}{r}$$

~~If we~~ So the theorem holds if

$\frac{1}{r_0} = \frac{1}{r_1} + \dots + \frac{1}{r_m}$. If $\frac{1}{r_0} > \frac{1}{r_1} + \dots + \frac{1}{r_m}$,

then we can decrease $r_1$ to some $r_1'$ so that

$$\frac{1}{r_0} = \frac{1}{r_1'} + \dots + \frac{1}{r_m}$$

and get (from Hölder)

$$M_{r_0}(\vec{a}_1 \cdots \vec{a}_m) \leq M_{r_1'}(\vec{a}_1) \cdots M_{r_m}(\vec{a}_m)$$

and then use the theorem of the means to conclude $M_{r_1'}(\vec{a}_1) \leq M_{r_1}(\vec{a}_1)$ and so obtain

$$M_{r_0}(\vec{a}_1 \cdots \vec{a}_m) \leq M_{r_1}(\vec{a}_1) \cdots M_{r_m}(\vec{a}_m).$$

This gives us the "if". For the "only if", set all the $\vec{a}_i = (1,0,..,0)$. Then

$$M_{r_0}(\vec{a}_1 \cdots \vec{a}_m) = \left( \frac{1^{r_0} + 0^{r_0} + \cdots + 0^{r_0}}{n} \right)^{1/r_0} = n^{-1/r_0}$$

and the rhs becomes

$$M_{r_1}(\vec{a}_1) \cdots M_{r_m}(\vec{a}_m) = n^{-1/r_1} \cdots n^{-1/r_m} = n^{-(1/r_1 + \cdots + 1/r_m)}$$

and it's clear that

$$M_{r_0}(\vec{a}_1 \cdots \vec{a}_m) \leq M_{r_1}(\vec{a}_1) \cdots M_{r_m}(\vec{a}_m)$$

becomes

$$n^{-1/r_0} \leq n^{-(1/r_1 + \cdots + 1/r_m)}$$

which is true only if $\frac{1}{r_0} \geq \frac{1}{r_1} + \cdots + \frac{1}{r_m}$. $\square$