Taylor's Theorem for Numerics.

In math 3100, we took a deep dive into the question

When does a series $\sum_{i=0}^{\infty} a_i$ converge to a limiting value s?

The final example of that class was the Taylor series

$$f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \dots + \frac{1}{n!}f^{(n)}(x)h^n + \dots$$

which we can write as

$$\sum_{i=0}^{\infty} \frac{1}{i!} f^{(i)}(x) h^i$$

as long as we ~~m~~ remember the conventions:

$$0! = 1, \quad n! = n(n-1)(n-2)\cdots 2\cdot 1$$

$$f^{(i)}(x) = \frac{d^i}{dx^i} f(x), \text{ the } i\text{th derivative}$$

$$f^{(0)}(x) = f(x)$$

$$h^0 = 1 \quad (\text{for all } h, \text{ including } 0^0 = 1).$$

We will rarely think about convergence of the entire Taylor series $\sum_{i=0}^{\infty} \frac{1}{i!} f^{(i)}(h) h^i$. Instead, we will focus on

$$P_K(h) = f(x) + f'(x)h + \cdots + \frac{f^{(K)}(x)}{K!} h^K$$

which we call the "K-th order" or "degree K" Taylor polynomial.

The Taylor polynomial $P_K(h)$ is a good approximation of f near x in the following sense:

Taylor's Theorem. (First form)

Let $R_K(h) = f(x+h) - P_K(h)$. (We call this the <u>remainder</u> term.) ~~Then~~ If f is K-times differentiable at x, then $\lim\limits_{h \to 0} \frac{R_K(h)}{h^K} = 0$.

Example. Let $f(x) = \sin x$ and compute $P_3(h)$ for $x = 1$.

We Know

$$f'(x) = \cos x$$
$$f''(x) = -\sin x$$
$$f'''(x) = -\cos x$$

We can then compute

$$P_3(h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \frac{f'''(x)}{6}h^3$$

$$= (\sin 1) + (\cos 1)h - \left(\frac{\sin 1}{2}\right)h^2 - \left(\frac{\cos 1}{6}\right)h^3$$

$$\approx 0.841 + 0.540\,h - 0.420\,h^2 - 0.090\,h^3$$

where the $\approx$ is only there because I rounded the sin 1 and cos 1 terms.

Taylor's theorem says that because $\sin x$ is differentiable 3 times at 1, we know

$$R_3(h) = f(x+h) - P_3(h)$$

$$= \sin(1+h) - \left(\sin 1 + \cos 1\, h - \frac{\sin 1}{2}h^2 - \frac{\cos 1}{6}h^3\right)$$

has the property

$$\lim_{h \to 0} \frac{R_3(h)}{h^3} = 0.$$

We can check this numerically with a quick Mathematica calculation.

Notes:

Since $h < 1$, $h > h^2 > h^3 > h^4$.

So Taylor's theorem also implies

$$\lim_{h \to 0} \frac{R_3(h)}{h} = \lim_{h \to 0} \frac{R_3(h)}{h^2} = 0$$

but it does not imply anything about

$$\lim_{h \to 0} \frac{R_3(h)}{h^4} = ?$$

Definition. We say that $f(x)$ is $o(g(x))$ "little-o" of $g(x)$ ~~at~~ as $x \to 0$ if

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = 0.$$

The "little-o" notation means that $f(x)$ is smaller than $g(x)$ near 0.

We can write Taylor's theorem more compactly as

$$f(x+h) = P_k(h) + o(h^k)$$
$$= f(x) + hf'(x) + \dots + \frac{f^{(k)}(x)}{k!}h^k + o(h^k).$$

where you read "$+o(h^k)$" as "plus some function of h which is $o(h^k)$".

Now if "$f(x)$ is k-times differentiable at x" is all we know, this is the most we can say.

Taylor's Theorem (Second form)

Suppose $f(x)$ is $K+1$ - times differentiable on $(x, x+h)$ and $f^{(K)}(x)$ is continuous on $[x, x+h]$. Then there exists some $\xi(h)$ in $(x, x+h)$ so that

$$R_K(h) = \frac{f^{(K+1)}(\xi(h))}{(K+1)!} h^{K+1}$$

That is

$$f(x+h) = f(x) + f'(x)h + \dots + \frac{f^{(K)}(x)}{K!}h^K + \frac{f^{(K+1)}(\xi(h))}{(K+1)!}h^{K+1}$$

this is not an approximation

Example. Since $\sin x$ is 4-times differentiable on $(1,2)$ and $\frac{d^3}{dx^3}\sin x = -\cos x$ is continuous on $[1,3]$, there is some ξ in $(1,2)$ so that

$$\sin(2) = \sin 1 + \cos 1 \cdot 2 + \frac{\sin 1}{2} \cdot 2^2 - \frac{\cos 1}{6} 2^3 + \frac{\sin \xi}{24} 2^4$$

This is a weird and kind of amazing statement about $\sin x$, but don't be distracted - Taylor says that a corresponding statement is true for <u>any</u> 4-times differentiable function with a continuous 3rd derivative.

Taylor Remainder Estimate.

If $f(x)$ is $(k+1)$-times differentiable on $(x, x+h)$ and $f^{(k)}$ is continuous on $[x, x+h]$ <u>and</u> $m \leq f^{(k+1)}(x) \leq M$ on $(x, x+h)$

then $\frac{m}{(k+1)!}h^{k+1} \leq R_k(h) \leq \frac{M}{(k+1)!}h^{k+1}$.

Equivalently,

$$P_k(h) + \frac{m}{(k+1)!}h^{k+1} \leq f(x+h) \leq P_k(h) + \frac{M}{(k+1)!}h^{k+1}$$

This theorem gives you quite a bit of control over f, assuming you know something about its derivatives.