Math 3500 Homework: Topology of $\mathbb{R}^n$.

This homework assignment covers Section 2.2 from Shifrin. Please print out this template[a] and use it to submit your work, putting each answer in the corresponding work area for the problem.

1. Use a dark (#2) pencil or a blue or black pen to do your work.
2. To help preserve blind grading, please *don't* write your name on the homework.
3. Some of the questions have a "final answer" area. If one is provided, please use it – it helps us group correct answers (and common mistakes) together in Gradescope.

1acdefk, 3, 4, 7, 8ab, 8cde*

1. (c.f. Shifrin 2.2.1) For each of the sets below, decide whether it is open, closed, or neither and prove your answer.

   (1) (5 points)

$$\{ x \in \mathbb{R} \mid 0 < x \leq 2 \}$$

Solution:

[a] There are printer kiosks around campus that you can use with your student ID if you need to. See this link for details.

(2) (5 points)

$$\left\{\, x \in \mathbb{R} \;\middle|\; x = 2^{-k} \text{ with } k \in \{0, 1, 2, \ldots ...\} \,\right\}$$

Solution:

(3) (5 points)

$$\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; y > 0 \right\}$$

Solution:

(4) (5 points)

$$\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; y \geq 0 \right\}$$

Solution:

(5) (5 points)

$$\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; y > x \right\}$$

Solution:

(6) (5 points)

$$\mathbb{Q} = \left\{ \frac{p}{q} \;\middle|\; p, q \text{ are integers} \right\} = \text{the rational numbers}$$

Solution:

2. (10 points) (Shifrin 2.2.4) Prove that the rectangle

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$$

is closed.

Solution:

3. (Shifrin 2.2.7) Suppose that $U, V \subset \mathbb{R}^n$ are open sets.

(1) (10 points) Prove that

$$U \cup V = \{ \mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \in U \text{ or } \mathbf{x} \in V \}$$

is open.

Solution:

(2) (10 points) Prove that

$$U \cap V = \{ \mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \in U \text{ and } \mathbf{x} \in V \}$$

is open.

Solution:

4. (c.f. Shifrin 2.2.8)

Definition. *Let* $S \subset \mathbb{R}^n$*. We say that* $\mathbf{a} \in S$ *is an* interior point *of* S *if there is some* $r > 0$ *so that* $B(\mathbf{a}, r) \subset S$*. We say that* $\mathbf{a} \in S$ *is a* boundary point[b] *of* S *if every* $B(\mathbf{a}, r)$ *with* $r > 0$ *contains points in* S *and points in* $\mathbb{R}^n - S$*.*

Definition. *The* closure $\bar{S}$ *of* $S \subset \mathbb{R}^n$ *is the intersection of all closed sets containing* S*:*

$$\bar{S} = \bigcap_{\substack{S \subset C \\ C \text{ closed}}} C$$

Equivalently, $\bar{S}$ *is the smallest closed set containing* S*.*

(1) (10 points) Show that every point of S is either an interior point or a boundary point.

Solution:

[b] Shifrin calls this a *frontier point*.

(2) (10 points) Give an example of a set $S \subset \mathbb{R}^n$ where every $\mathbf{a} \in S$ is a boundary point.

Solution:

(3) (10 points) Prove that the set ∂S of boundary points of S is closed.

Solution:

(4) (10 points) Prove that $S \cup \partial S$ is closed.

Solution:

(5) (10 points) Suppose that $S \subset C$ and C is closed. Show that $(S \cup \partial S) \subset C$.[c] Use this to prove that the closure $\bar{S} = S \cup \partial S$.[d]

Solution:

[c]Hint: Prove that $\mathbb{R}^n - C \subset \mathbb{R}^n - (S \cup \partial S)$.

[d]To prove that two sets A and B are the same, one usually proves that $A \subset B$ and $B \subset A$.