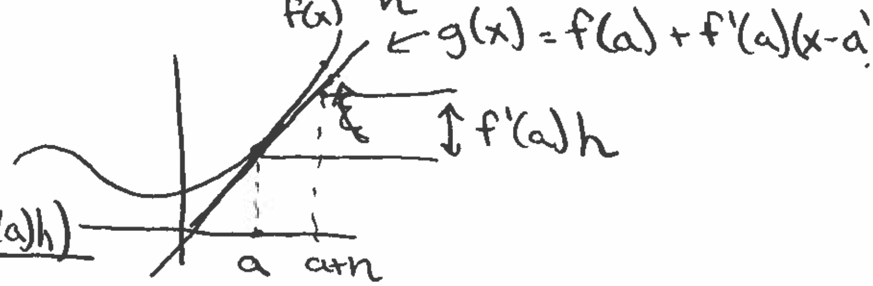
Differentiability.

Definition. $f'(a) = \lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$, if this limit exists.

Redefinition. $f'(a) = m \iff \lim_{h\to 0} \dfrac{f(a+h)-f(a)-mh}{h} = 0$

Prove these are same.



$$\frac{\text{Error}}{h} = \frac{f(a+h) - (f(a)+f'(a)h)}{h}$$

The derivative makes $\lim_{h\to 0} \dfrac{\text{Error}(h)}{h} = 0$.

Definition. Let $U \subset \mathbb{R}^n$ be open and $\vec{a} \in U$. A function $\vec{f}: U \to \mathbb{R}^m$ is <u>differentiable at $\vec{a}$</u> if there exists a linear map $Df(\vec{a}): \mathbb{R}^n \to \mathbb{R}^m$ so

$$\lim_{\vec{h}\to 0} \frac{\vec{f}(\vec{a}+\vec{h}) - \vec{f}(\vec{a}) - (Df(\vec{a}))(\vec{h})}{\|\vec{h}\|} = \vec{0}$$

This linear map is called the <u>differential</u> or <u>Jacobian</u>.

Definition. The <u>tangent plane</u> (or <u>tangent space</u>) of $\vec{f}(\vec{x})$ at $\vec{a}$ is the graph of $g(\vec{x}) = \vec{f}(\vec{a}) + Df(\vec{a})(\vec{x}-\vec{a})$

Proposition. If $\vec{f}: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\vec{a}$ then the partial derivatives $\dfrac{\partial f_i}{\partial x_j}$ exist and

$$[Df(\vec{a})] = \left[\frac{\partial f_i}{\partial x_j}(\vec{a})\right]$$