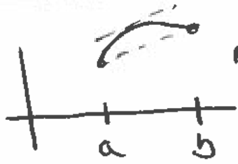
Higher partials.



mean value theorem

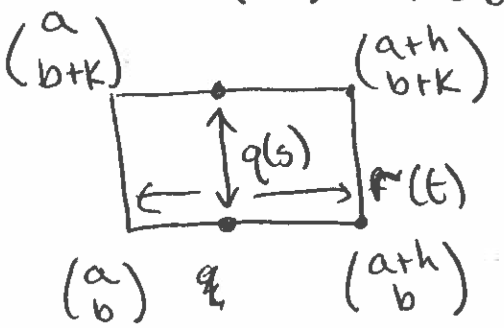
$f(b)-f(a) = f'(c)(b-a)$

for some $c \in (a,b)$

Theorem. If $\vec{f}: \mathcal{U} \to \mathbb{R}^m$ ($\mathcal{U} \subset \mathbb{R}^n$) is a C^2 function, then for any $i,j \in 1,\dots,n$ $\dfrac{\partial^2 \vec{f}}{\partial x_i \partial x_j} = \dfrac{\partial^2 f}{\partial x_j \partial x_i}$

Proof. Let $m=1, n=2, i=1, j=2$.

$$\Delta\begin{pmatrix}h\\k\end{pmatrix} = f\begin{pmatrix}a+h\\b+k\end{pmatrix} - f\begin{pmatrix}a+h\\b\end{pmatrix} - f\begin{pmatrix}a\\b+k\end{pmatrix} + f\begin{pmatrix}a\\b\end{pmatrix}$$



Now let $q(s) = f\begin{pmatrix}s\\b+k\end{pmatrix} - f\begin{pmatrix}s\\b\end{pmatrix}$.

$$\begin{aligned}\Delta\begin{pmatrix}h\\k\end{pmatrix} &= q(a+h) - q(a)\\ &= h\,q'(\xi), \text{ for some } \xi \text{ in } [a, a+h]\\ &= h\left(\frac{\partial f}{\partial x}\begin{pmatrix}\xi\\b+k\end{pmatrix} - \frac{\partial f}{\partial x}\begin{pmatrix}\xi\\b\end{pmatrix}\right)\\ &= hk\,\frac{\partial^2 f}{\partial y \partial x}\begin{pmatrix}\xi\\\eta\end{pmatrix} \text{ for some } \eta \text{ in } [b, b+k]\end{aligned}$$

Similarly, if

$r(t) = f\begin{pmatrix}a+h\\t\end{pmatrix} - f\begin{pmatrix}a\\t\end{pmatrix}$

then

$\Delta\begin{pmatrix}h\\k\end{pmatrix} = hk\,\dfrac{\partial^2 f}{\partial x \partial y}\begin{pmatrix}\sigma\\\gamma\end{pmatrix}$

for some $\sigma \in (a, a+h)$, $\gamma \in (b, b+k)$. Thus for any h, k

$$\frac{\partial^2 f}{\partial x \partial y}\begin{pmatrix}\sigma\\\gamma\end{pmatrix} = \frac{\partial^2 f}{\partial y \partial x}\begin{pmatrix}\xi\\\eta\end{pmatrix}$$

Since these are continuous functions, taking the limit as $h,k \to 0$, gives us $\dfrac{\partial^2 f}{\partial x \partial y}\begin{pmatrix}a\\b\end{pmatrix} = \dfrac{\partial^2 f}{\partial y \partial x}\begin{pmatrix}a\\b\end{pmatrix}$. $\square$

Examples. Harmonic function. Laplacian.

Wave equation $\dfrac{\partial^2 f}{\partial t^2} = c^2 \dfrac{\partial^2 f}{\partial x^2}$

$$\cancel{f(x+ct} \quad f\begin{pmatrix}x\\t\end{pmatrix} = \varphi(x+ct) + \psi(x-ct)$$