Quadratic forms and the second derivative test

Recall.

If f is C^2 in $(a-r, a+r)$ for $r>0$ and $f'(a)=0$, then

$$f''(a) > 0 \quad \Rightarrow \quad a \text{ is a local min}$$
$$f''(a) < 0 \quad \Rightarrow \quad a \text{ is a local max}$$

($f''(a) = 0$ doesn't rule out anything)

Why?

Lemma. Suppose $g: [0,1] \to \mathbb{R}$ is C^2. Then

$$g(1) = g(0) + g'(0) + \tfrac{1}{2} g''(\xi)$$

xi

for some $\xi \in [0,1)$.

Proof. (Clever!) Let

$$P(t) = g(0) + g'(0)t + Ct^2$$

observe that if

$$C = g(1) - g(0) - g'(0)$$

we have

$$P(1) = g(1)$$
$$P(0) = g(0)$$
$$P'(0) = g'(0)$$

So consider $h(t) = g(t) - P(t)$. We have

$$h(0) = 0, \quad h(1) = 0$$

so (Rolle's theorem) $\exists$ some $c \in (0,1)$ so that $h'(c) = 0$. Now

$$h'(0) = 0, \quad h'(c) = 0$$

so (Rolle's theorem) $\exists$ some $\xi \in (0,c)$ so that $h''(\xi) = 0$.

Now

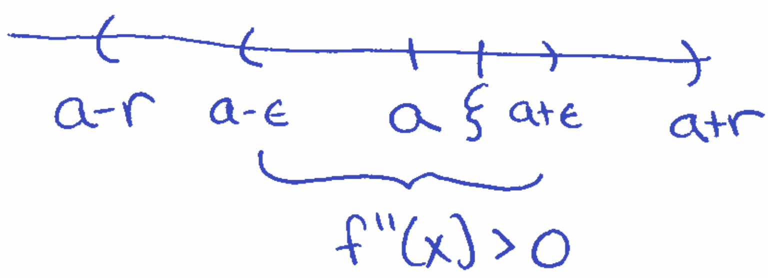
$$0 = h''(\xi) = g''(\xi) - P''(\xi)$$
$$= g''(\xi) - 2C$$

so $g''(\xi) = 2C$. But then

$$P(t) = g(0) + g'(0)t + \frac{t^2}{2} g''(\xi).$$

and $P(1) = g(1) = g(0) + g'(0)\cancel{\xi} + \frac{1}{2} g''(\xi).$

This is the proof of the 2nd derivative test: If $f''(a) > 0$, then (by continuity) $\exists$ some $\epsilon > 0$ so~~me~~ that $f''(x) > 0$ on $(a-\epsilon, a+\epsilon)$.



At any $y \in (\cancel{a-\epsilon}\; a, a+\epsilon)$, $\exists$ some $\xi \in (a, y)$ so that

$$f(y) = f(a) + f'(a)(y-a) + \frac{1}{2} f''(\xi)(y-a)^2$$

$$= f(a) + 0 + \frac{1}{2}\underbrace{f''(\xi)}_{>0}\underbrace{(y-a)^2}_{>0}$$

So $f(y) > f(a)$. ~~Thus~~

We now want to generalize.

Definition. Hessian matrix. Hess form.

Prop. Suppose $f: B(a,r) \to \mathbb{R}$ is C^2, for $\|\vec{h}\| < r$,

$$f(\vec{a}+\vec{h}) = f(\vec{a}) + Df(\vec{a})\vec{h} + \frac{1}{2}H_{\cancel{a}\,a+\xi h}(\vec{h})$$

for some $0 < \xi < 1$.

Definition. Pos def, neg def, psd, nsd, ind

Example. $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Prop. Suppose $f: B(a;r) \to \mathbb{R}$ is C^2, $Df(\vec{a}) = \vec{0}$

pos def $\to$ min, neg def $\to$ max

indef $\to$ saddle, (psd or nsd tells us nothing)

$AD - B^2 > 0$ $\quad$ $A > 0$ local min

$A < 0$ local max

$AD - B^2 < 0$ saddle

($AD - B^2 = 0$ tells us nothing).

Example. $x^3 + y^2 - 6xy$.

LU decomp. LDL^T decomp.

~ but more to come