The Transpose.

Definition. If $A \in Mat_{m\times n}$, with entries $A = [a_{ij}]$, then $A^T \in Mat_{n\times m}$ is the matrix where $(A^T)_{ij} = A_{ji}$.

Definition. A is <u>symmetric</u> if $A = A^T$ and <u>skew-symmetric</u> if $A = -A^T$.

Note. Symmetric and skew-symmetric matrices are always <u>square</u> or $n\times n$ matrices.

Proposition. If $A, A' \in Mat_{m\times n}$ and $B \in Mat_{n\times p}$ and $c \in \mathbb{R}$,

1. $(A^T)^T = A$
2. $(cA)^T = cA^T$
3. $(A+A')^T = A^T + (A')^T$
4. $(AB)^T = B^T A^T$

Proof. We will only prove 4.

Suppose $A = [a_{ij}]$ and $A^T = [(a^T)_{ij}]$, where $a_{ij} = (a^T)_{ji}$, and similarly $B = [b_{ij}]$ and $B^T = [(b^T)_{ij}]$, $b_{ij} = (b^T)_{ji}$.

Now

$$
\begin{aligned}
\left((AB)^T\right)_{ij} &= (AB)_{ji} \\
&= \sum_K a_{jK} b_{Ki} \\
&= \sum_K (a^T)_{Kj} (b^T)_{iK} \\
&= \sum_K (b^T)_{iK} (a^T)_{Kj} \\
&= (B^T A^T)_{ij}.
\end{aligned}
$$

$\square$

Lemma. If we write $\vec{v}, \vec{w} \in \mathbb{R}^n$ as $n \times 1$ matrices $v, w \in \mathrm{Mat}_{n\times 1}$, then

$$v^T w = [\vec{v} \cdot \vec{w}] \in \mathrm{Mat}_{1\times 1}$$

Proof. Follows directly from our definition of matrix multiplication. $\square$

Proposition. If $A \in \mathrm{Mat}_{m\times n}$, $\vec{x} \in \mathbb{R}_n$ and $\vec{y} \in \mathbb{R}^m$,

$$A\vec{x} \cdot \vec{y} = \vec{x} \cdot (A^T \vec{y}).$$

Proof.

$$[A\vec{x} \cdot \vec{y}] = (Ax)^T y = x^T A^T y = [\vec{x} \cdot A^T \vec{y}].$$

$\boxed{\text{This is really important!}}$

The structural similarity between "transpose" and "inverse" makes you think that there's some connection.

Definition. A matrix $A \in Mat_{n \times n}$ is said to be <u>orthogonal</u> if $A^T = A^{-1}$.

Proposition. $A \in Mat_{n \times n}$ is orthogonal if and only if $A\vec{x} \cdot A\vec{y} = \vec{x} \cdot \vec{y}$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$.

Proof. ($\Rightarrow$) Suppose A is orthogonal. Then

$$\begin{aligned} A\vec{x} \cdot A\vec{y} &= \cancel{A}\ \vec{x} \cdot A^T A \vec{y} \\ &= \vec{x} \cdot A^{-1} A \vec{y} \\ &= \vec{x} \cdot I \vec{y} = \vec{x} \cdot \vec{y}. \end{aligned}$$

($\Leftarrow$) Suppose $A\vec{x}\cdot A\vec{y} = \vec{x}\cdot\vec{y}$ for all $\vec{x},\vec{y}\in\mathbb{R}^n$. In particular, this means

$$\vec{x}\cdot\vec{y} = A\vec{x}\cdot A\vec{y} = \vec{x}\cdot A^T A\vec{y}$$

Now suppose $\vec{x}=\vec{e}_i$ and $\vec{y}=\vec{e}_j$. Then $A^TA\vec{e_j} = A^TA\vec{e}_j$ = the jth column of A^TA. Further, $\vec{x}\cdot A^TA\vec{y} = \vec{e}_i\, A^TA\, e_j$ = the ith entry in the jth column, or $(A^TA)_{ij}$

Thus

$$(A^TA)_{ij} = \vec{e}_i\cdot\vec{e}_j$$

But $\vec{e}_i\cdot\vec{e}_j = 0$ if $i\neq j$ and 1 if $i=j$. So this proves that $A^TA = I$ and (by uniqueness of inverses) that $A^T = A^{-1}$. $\square$