Math 4250/6250 Exam #2

This take-home exam covers the material on surfaces through the Theorema Egregium (Gauss curvature is an isometry invariant). This material is from DoCarmo Chapter 2, Sections 1 though 5, Chapter 3, Sections 1 through 3, and Chapter 4, Section 1. The corresponding posted homework notes are #7 through #17. Please don't be afraid to read over the notes and these sections in the book as you work on the problems– there is more in the notes than we were able to cover in the lectures, and some of those extra facts might be helpful to you as you work on the exam problems.

As before, please pick 4 of the following problems. If you are a graduate student, you must attempt a challenge problem. If you are an undergraduate, you can do a challenge problem as one of your four problems (and will get bonus credit if you get it right).

You **are** permitted to use	You **are not** permitted to use
Maple (or Mathematica or MATLAB)	The internet
A calculator (or graphing calculator)	
DoCarmo	Other books
Your notes	Other people's notes
Your brain	Other people's brains
Class notes posted on the website	

1. A *loxodrome* is a curve $\alpha(t) = (\theta(t), \phi(t))$ on the unit sphere which makes a constant angle α with each meridian curve $\beta(t) = (\theta_0, t)$. Consulting section #10 of the class lecture notes, determine the length of a loxodrome which starts at the equator and ends at the north pole.

   Your answer should be a function of the angle α. We will say that the equator itself is a loxodrome of angle $\alpha = \pi/2$, while a meridian is a loxodrome of angle 0. For full credit, actually do the integral. For partial credit, just (correctly) setting up the length integral is ok.

2. A point $p \in \mathbf{R}^3$ is on the surface S if p is on any tangent line to a curve $\alpha(s)$. Please assume that α is parametrized by arclength.

   Find a parametrization for the surface S in the form $x(u, v) = (x(u, v), y(u, v), z(u, v))$ and compute the coefficients E, F, and G of the first fundamental form of the surface. Your answer should be a function of α and its derivatives.

3. A certain surface is given by

$$x(u, v) = \left(u, v, \frac{\ln(\cos(av)\sec(au))}{a} \right).$$

   Compute the mean curvature of this surface. For extra credit, graph the surface.

4. Compute the Christoffel symbols for the surface $x(u, v) = (u \cos v, u \sin v, 0)$. Notice that this surface is the x-y plane. Compute the Gaussian curvature of this surface from the Christoffel symbols.

5. **(Not all that hard) Challenge Problem for Graduate Students.** The cone surface over a space curve $\alpha(s)$ is defined by

$$x(u, v) = u\alpha(v),$$

where $u > 0$. Find conditions on α which guarantee that the surface is regular. Then compute the Gaussian and Mean curvatures of this surface.

6. **Harder Challenge Problem for Graduate Students.** The first fundamental form of a surface is defined by $I_p(v) = \langle dx(v), dx(v) \rangle$, where $\langle , \rangle$ is the usual dot product in $\mathbf{R}^3$. The second fundamental form is defined by $II_p(v) = \langle dN(v), dx(v) \rangle$. One can define a **third** fundamental form by $III_p(v) = \langle dN(v), dN(v) \rangle$. Show that the three fundamental forms of the surface obey the relation
$$K\, I_p(v) - 2H\, II_p(v) + III_p(v) = 0.$$