Math 4250/6250 Homework #1

This homework assignment covers the notes 1-5. (Review of Linear Algebra, Arclength and Rectifiability, Framed Curves and the Frenet Frame, More on Curvature and Torsion, and The Bishop Frame). Until we are used to distinguishing between vector and scalar-valued functions, we're going to write vectors with an arrow over them ($\vec{\alpha}$) and leave scalar variables alone (s).

1. (Derivatives and inverse functions.) Suppose that $f(x)$ is a function, and $g(x)$ is its inverse function; that is, that
$$f(g(x)) = x,$$
Find a relationship between f' and g' by differentiating both sides of the equation above.

2. (Reparametrizing by arclength.) Suppose that $\vec{\alpha}(t) : \mathbf{R} \to \mathbf{R}^3$ is a space curve. Let $s(t)$ be the arclength function
$$s(t) = \int_0^t |\vec{\alpha}'(x)| \, dx.$$
   a. Compute the derivative $\frac{d}{dt}s(t)$ using the Fundamental Theorem of Calculus.

   b. Suppose that $i(t)$ is the inverse function of $s(t)$, so that $i(s(t)) = t$. Compute $\frac{d}{dt}i(t)$ using the above (and the first problem).

   c. Suppose that we let $\beta(t) = \alpha(i(t))$. Compute $\left|\frac{d}{dt}\beta(t)\right|$ and conclude that $\beta(t)$ is parametrized by arclength.

3. (Hyperbolic trig functions) We are used to the usual trigonometric functions $\sin(x)$ and $\cos(x)$. We're now going to introduce two new functions: $\sinh(x)$ and $\cosh(x)$, defined by
$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}.$$
   a. Prove that $\cosh(0) = 1$ and $\sinh(0) = 0$.

   b. Prove that $\frac{d}{dx}\sinh(x) = \cosh(x)$ and $\frac{d}{dx}\cosh(x) = +\sinh(x)$. (Notice that this is *different* from the regular trig functions.)

   c. Prove that $\cosh^2(x) - \sinh^2(x) = 1$ for all x.

   d. Use Mathematica or Desmos to graph $\cosh x$ and $\sinh x$ for x between 0 and 4.

   e. Use Mathematica[1] or Desmos[2] to plot the parametrized curves
$$\vec{\alpha}(t) = (\cos t, \sin t), \quad \vec{\beta}(t) = (\cosh t, \sinh t)$$
   for $-10 \leq t \leq 10$. Can you recognize these curves? Why are $\cosh t$ and $\sinh t$ called *hyperbolic* trigonometric functions?

[1] Use `ParametricPlot`.

[2] Any expression in the form $(x(t), y(t))$ will be plotted as a parametric curve in Desmos, *but only if t is the variable!*.

4. (Derivatives of Inverse Trig and Hyperbolic Trig Functions) Recall that the inverses of the trig (and, as it turns out, the hyperbolic trig) functions are preceded by "arc"[3]. This means that

$$\sin(\arcsin x) = x, \quad \cos(\arccos x) = x, \quad \sinh(\operatorname{arcsinh} x) = x, \quad \cosh(\operatorname{arccosh} x) = x.$$

We can compute the derivatives of the inverse trig functions and inverse hyperbolic trig functions by using the inverse function/chain rule trick from Problem 1. For example:

$$\begin{aligned} \frac{d}{dx}\sin(\arcsin x) &= \frac{d}{dx}x \\ \cos(\arcsin x)\frac{d}{dx}\arcsin x &= 1 \\ \frac{d}{dx}\arcsin x &= \frac{1}{\cos(\arcsin x)}. \end{aligned}$$

At this point, we remember that $\cos^2 x + \sin^2 x = 1$, so $\cos x = \sqrt{1-\sin^2 x}$ and so we can continue:

$$\begin{aligned} \frac{d}{dx}\arcsin x &= \frac{1}{\sqrt{1-\sin^2(\arcsin x)}} \\ \frac{d}{dx}\arcsin x &= \frac{1}{\sqrt{1-x^2}}. \end{aligned}$$

a. For practice, repeat the above derivation to figure out $\frac{d}{dx}\arccos x$.

b. Make a similar argument to find out $\frac{d}{dx}\operatorname{arcsinh} x$ and $\frac{d}{dx}\operatorname{arccosh} x$.

c. Now integrate both sides to express these derivatives as integrals. Hang on to them; you'll need them in a problem or two!

5. (Autonomous differential equations) Recall that a first order differential equation for an unknown function $u(t)$ is an equation in the form

$$u'(t) = F(u(t), t)$$

(where F is some function of $u'(t)$ and t) and that the equation is *autonomous* if the right hand side can be written only in terms of $u(t)$ as

$$(\star) \qquad u'(t) = F(u(t))$$

[3] Why? Well $\arcsin x$ is the angle θ for which $\sin\theta = x$. And if you measure θ in radians, the angle θ is the angle covered by an arc of length θ on the unit circle. So $\arcsin x$ literally means "the arc whose sine is x".

As you learned in MATH 2700, every autonomous first-order ODE may be solved as follows

$$u'(t) = F(u(t))$$
$$\frac{u'(t)}{F(u(t))} = 1$$
$$\int \frac{u'(t)}{F(u(t))}\,dt = \int 1\,dt$$
$$\int \frac{1}{F(u)}\,du = t + k$$
$$G(u) = t + k$$

where $G(u)$ is any antiderivative of $\frac{1}{F(u)}$. If we can find an inverse function $G^{-1}(u)$ for $G(u)$, we can solve explicitly for u as

(♠) $$u(t) = G^{-1}(t+k)$$

This is the most general solution to equation ($\star$). If we know some value $u(t_0) = u_0$ we can plug it in to (♠) to solve for k.

a. Use the method above to find the most general solution to
$$u'(t) = u(t)^2.$$

b. Suppose $u(t_0) = u_0$. Plug it in to the results of (a) to write down the general solution in terms of t_0 and u_0.

c. Suppose $t_0 = 0$ and $u_0 = 2$. Plug in to the results of (b) to write down the particular solution with $u(0) = 2$.

6. (The Caternary Curve) Suppose we parametrize the graph of a function $f(x)$ by $\vec{\alpha}(t) = (t, f(t))$.
   a. Prove that the arclength formula is given by
   $$s(t) = \int_0^t \sqrt{1 + (f'(x))^2}\,dx.$$

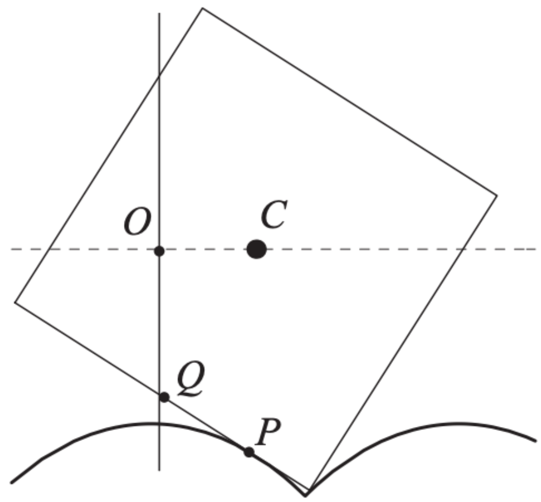
   b. Show that if $f(x) = \cosh x$ then $s(t) = \sinh t$.

   c. Reparametrize the caternary curve $\alpha(t) = (t, \cosh t)$ by arclength, following the model you established in Problem 2. Try to simplify the results as much as possible using hyperbolic trig identities.

7. (The planar operator $\perp$) Suppose that $\vec{v} = (v_1, v_2)$ is vector in $\mathbb{R}^2$. We define the operator $\perp$ by $\vec{v}^\perp = (-v_2, v_1)$.
   a. Show that $\left\langle \vec{v}, \vec{v}^\perp \right\rangle = 0$, and hence that $\vec{v}$ and $\vec{v}^\perp$ are orthogonal to one another.

   b. Show that $\left\langle \vec{v}, \vec{w}^\perp \right\rangle = |\vec{v}|\,|\vec{w}| \sin\theta$, where θ is the angle between $\vec{v}$ and $\vec{w}$.

8. (The square-wheeled car) Consider the figure below, where a square with sidelength 2 is rolling along a bumpy road.



We will parametrize the progress of the square by the arclength s it has rolled along the road since starting in the horizontal position with center $C(0) = O$. Assume that the square rolls smoothly, so that as a curve,

$$\vec{C}(s) = (c_1(s), c_2(s)) = (c_1(s), 0)$$

Assume that $\vec{\alpha}(s)$ is an arclength parametrization of the road, so that $\vec{P}(s) = \vec{\alpha}(s) = (x(s), y(s))$, and assume that $\vec{\alpha}(0) = (0, -1)$. Further, assume that $\vec{Q}(s)$ is always the midpoint of that edge of the square. Since the square has rolled without slipping along the portion of the edge between $\vec{Q}(s)$ and $\vec{P}(s)$, we know that $\left|\vec{Q}(s) - \vec{P}(s)\right| = s$, and that the edge of the square is tangent to $\alpha(s)$ at the point of contact $\vec{P}(s)$. Assume that $\vec{O}(s) = \vec{0}$ is the origin for all t.

Note: The parameter s is an arclength parameter for $\vec{\alpha(s)} = \vec{P}(s)$ (only). That is, $\vec{O}(s)$, $\vec{Q}(s)$, and $\vec{C}(s)$ certainly trace out parametrized curves in the plane as the square wheel turns. But we don't have any reason to believe that *those* curves are arclength parametrized.

a. Write down a formula for $\vec{P}(s) - \vec{O}(s)$ in terms of $x(s)$, $y(s)$, and s.

b. Write down a formula for $\vec{Q}(s) - \vec{P}(s)$ in terms of $x(s)$, $y(s)$, and s.

c. Write down a formula for $\vec{C}(s) - \vec{Q}(s)$ in terms of $x(s)$, $y(s)$, and s.

d. Add the results of (a)-(c) to find a formula for $\vec{C}(s) - \vec{O}(s) = (c_1(s), c_2(s))$ in terms of $x(s)$, $y(s)$ and s. Compute the derivative $c_2'(s) = 0$ to get a relationship between s, $x''(s)$, and $y''(s)$.

e. Remember that $1 = |\vec{\alpha}'(s)|^2 = x'(s)^2 + y'(s)^2$. Differentiate both sides with respect to s to get a relationship between $x'(s)$, $x''(s)$, $y'(s)$ and $y''(s)$.

f. Combine the results of (d) and (e) to get a relationship between s, $x'(s)$, and $y'(s)$.

g. Now there is some function f so that $y(s) = f(x(s))$. Differentiating both sides with respect to s), we know that

$$y'(s) = f'(x(s))x'(s)$$

Use this and the results of (f) to find a formula for $f'(x)$ in terms of s.

h. Note that we can reparametrize $\vec{\alpha}(s)$ as $\vec{\beta}(t) = (t, f(t))$ as we did in (4). Further, remember that s is the arclength along $\vec{\beta}(t)$ from 0 to x. Thus

$$s(x) = \int_0^x \left|\vec{\beta}'(t)\right| \, dt$$

and so

$$s'(x) = \left|\vec{\beta}'(x)\right|$$

Use the parametrization for $\beta(x)$ to write a formula for $s'(x)$ in terms of $f'(x)$.

i. Differentiate the results of (g) by x (on both sides) to get an expression for $f''(x)$ in terms of $s'(x)$, and use the results of (h) to write $f''(x) = F(f'(x))$ for some function F. Solve this autonomous differential equation for $f'(x)$ using the technique you reviewed in Problem 5 and the fact that you know $f'(0)$. Remember Problem 4.

j. Integrate your formula for $f'(x)$ to get $f(x)$. Use the fact that you know $f(0)$ to eliminate the constant of integration.

9. (Frenet Apparatus) Find the Frenet frame $\vec{T}(s)$, $\vec{N}(s)$, $\vec{B}(s)$, $\kappa(s)$ and $\tau(s)$ for the arclength-parametrized curve

$$\vec{\alpha}(s) = \left(\frac{1}{3}(1+s)^{3/2}, \frac{1}{3}(1-s)^{3/2}, \frac{1}{\sqrt{2}}s\right).$$

10. (The Darboux Vector) If $\gamma(s)$ is an arclength-parametrized curve with nonzero curvature, find a vector $\omega(s)$, expressed as a linear combination of T, N, and B so that

$$T'(s) = \omega(s) \times T(s)$$
$$N'(s) = \omega(s) \times N(s)$$
$$B'(s) = \omega(s) \times B(s)$$

This vector is called the *Darboux vector*. Find a formula for the length of the Darboux vector in terms of the curvature $\kappa(s)$ and torsion $\tau(s)$ of the curve.

Hint: Any vector $\omega(s)$ can be written as a linear combination of the vectors $T(s)$, $N(s)$, and $B(s)$ with coefficients $a(s)$, $b(s)$ and $c(s)$ which are (scalar) functions of s because $T(s)$, $N(s)$ and $B(s)$ always form an orthonormal basis for $\mathbf{R}^3$ (regardless of s). That is,

$$\vec{\omega}(s) = a(s)\vec{T}(s) + b(s)\vec{N}(s) + c(s)\vec{B}(s).$$

So really the problem is to figure out the functions $a(s)$, $b(s)$ and $c(s)$.

11. (The curvature of the graph of a function) Prove that a plane curve $\vec{\alpha}(t) = (t, f(t), 0)$ has curvature

$$\kappa(t) = \frac{|f''(t)|}{(1 + f'(t)^2)^{3/2}}$$

You'll need the formula

$$(\clubsuit) \qquad \kappa(t) = \frac{|\vec{\alpha}'(t) \times \vec{\alpha}''(t)|}{|\vec{\alpha}'(t)|^3}$$

from the notes "More on Curvature and Torsion".

12. (The torsion of non arclength-parametrized curve) Reread the proof of (♣) in the notes "More on Curvature and Torsion". Now make the same kind of argument to show that

$$\tau(t) = \frac{\langle \vec{\alpha}'(t), \vec{\alpha}''(t) \times \vec{\alpha}'''(t) \rangle}{|\vec{\alpha}'(t) \times \vec{\alpha}''(t)|}$$

for any space curve $\vec{\alpha}(t)$, regardless of whether or not $\vec{\alpha}(t)$ is parametrized by arclength.

1. Challenge Problems

1. (The Shortest Curve Between Points) Let $\alpha\colon I \to \mathbf{R}^3$ be a a differentiable parametrized curve. Suppose $[a, b] \in I$ and $\alpha(a) = p$ while $\alpha(b) = q$.

   a. Show that for any constant vector v with $|v| = 1$,

   $$\langle q - p, v \rangle = \int_a^b \langle \alpha'(t), v \rangle \; dt \leq \int_a^b |\alpha'(t)| \, dt.$$

   b. Let

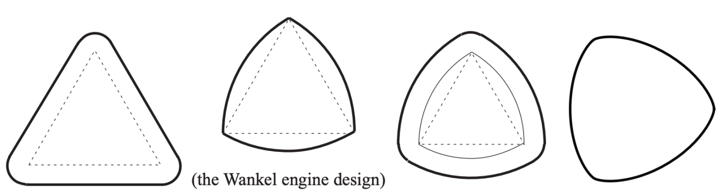
   $$v = \frac{q - p}{|q - p|}$$

   and show that

   $$|\alpha(b) - \alpha(a)| \leq \int_a^b |\alpha'(t)| \, dt.$$

   That is, the curve of shortest length joining two points is the straight line!

2. (Constant Breadth) A closed planar curve $\vec{\alpha}(s)$ is said to have constant breadth if the distance between parallel tangent line of $\vec{\alpha}(s)$ is always μ. A circle is an example of such a curve, but it's not the only one:



We'll assume that $\vec{\alpha}(s) : [0, L] \to \mathbf{R}^2$ is an arclength parametrization of $\vec{\alpha}(s)$ with $\vec{\alpha}(0) = \vec{\alpha}(L)$ (and all derivatives of $\vec{\alpha}(s)$ equal at 0 and L as well. We'll also assume that $\kappa(s) \neq 0$.

a. Let's call two points with parallel tangent lines *opposite* points. Suppose that $\vec{\beta}(s)$ is always the opposite point to $\vec{\alpha}(s)$. (Note that just because s is an arclength parameter for $\vec{\alpha}(s)$ doesn't mean that s is an arclength parameter for $\vec{\beta}(s)$; it's probably not.) Since the tangent and normal vectors $\vec{T}(s)$, $\vec{N}(s)$ at $\vec{\alpha}(s)$ are a basis for the plane, there are some coefficients $c_1(s)$ and $c_2(s)$ so that

$$\vec{\beta}(s) - \vec{\alpha}(s) = c_1(s)\vec{T}(s) + c_2(s)\vec{N}(s).$$

Prove that $c_2(s) = \mu$ and then prove that $c_1(s) = 0$. Conclude that the chord joining opposite points is normal to the curve at both ends.

b. Let $\kappa_\alpha(s)$ be the curvature of $\vec{\alpha}$ at s, and $\kappa_\beta(s)$ be the curvature of $\vec{\beta}$ at s. Prove that

$$\frac{1}{\kappa_\alpha(s)} + \frac{1}{\kappa_\beta(s)} = \mu$$

Hint: Let $\vec{T}_\beta(s)$ and $\vec{N}_\beta(s)$ denote the tangent and normal vectors to $\vec{\beta}(s)$. How are they related to $\vec{T}(s)$ and $\vec{N}(s)$? Can you use (♣) to compute $\kappa_\beta(s)$?