# Math 4250/6250 Homework #5

This homework assignment covers our notes on the Gauss map (11), the meaning of the Gauss map (12) and the second fundamental form (13). Please pick 4 of the following problems. Remember that undergraduate students should average **one** challenge problem per assignment, while graduate students should average **two** challenge problems per assignment.

## 1. Regular Problems

1. Show that at a hyperbolic point on a regular surface $S$, the principal directions bisect the asymptotic directions.
2. Let $\alpha(s)$ be a regular curve on a surface $S$. Suppose that at the point $\alpha(s)$, the surface $S$ has Gaussian curvature $K > 0$ and principal curvatures $k_1$ and $k_2$. Show that the curvature $\kappa(s)$ of $\alpha$ at this point satisfies
$$\kappa(s) \geq \min(|k_1|, |k_2|)$$
3. Suppose that $S$ is a surface with principal curvatures $k_1$ and $k_2$ which obey the inequalities $|k_1| \leq 1$ and $|k_2| \leq 1$. Is it true that the (space) curvature $\kappa$ of every curve $\alpha$ on $S$ also has $|\kappa| \leq 1$? (Be careful! And if you don't think so, give a specific example!)
4. Suppose that $\alpha(s)$ is an asymptotic curve of a surface $S$ with Gauss curvature $K$ and that the curvature of $\alpha(s)$ is not equal to zero. Prove that the torsion $\tau(s)$ of $\alpha$ is given by
$$|\tau(s)| = \sqrt{-K}.$$

## 2. Challenge Problems

1. Show that the mean curvature $H$ of a surface $S$ at a point $p \in S$ can be expressed as the average of normal curvatures of curves in $S$ through $p$. Suppose that $\kappa_n(\theta)$ is the normal curvature of a curve in $S$ in direction $\cos\theta\vec{x}_u + \sin\theta\vec{x}_v$. Then we must prove
$$H = \frac{1}{\pi}\int_0^{\pi} \kappa_n(\theta)\,\mathrm{d}\theta.$$
2. Suppose that $\vec{v}$ and $\vec{w}$ are orthogonal directions in the tangent plane $T_pS$ to a surface $S$. Show that the sum $\kappa_n(\vec{v}) + \kappa_n(\vec{w})$ does not depend on the choice of $\vec{v}$ and $\vec{w}$ as long as they are orthogonal.
3. Let $p$ be a hyperbolic point on a surface $S$. Fix any $\vec{v} \in T_pS$. Describe a procedure for finding the conjugate direction to $\vec{v}$ using the Dupin indicatrix.