Continuous Probability Distributions.

So far, we've discussed experiments where the sample space is finite. But what about more general cases?

Definition. Suppose that $\Omega \subset \mathbb{R}^n$, and we have a function $\mu : \Omega \to \mathbb{R}$ so that

1) $\mu(\omega) \geq 0$ for all $\omega \subset \Omega$.

2) $\int_{\Omega} \mu(\omega)\, d\omega = 1$.

We say that μ is a probability density function for a r.v. X with outcomes in Ω

if

$$P(X \in E) = \int_E \mu(\omega)\, d\omega$$

for every subset $E \subset \Omega$ for which the integral at right exists.

Note. These are <u>not</u> all subsets of Ω.

Example. We construct a spinner and take X as the angle (in radians) of the final position of the arrow.

$$\Omega = [0, 2\pi)$$

$$\mu(\omega) = \frac{1}{2\pi}$$

This choice of density function assumes that every outcome is equally likely.

For any interval $(a,b) \subset [0, 2\pi)$ of angles, we compute

$$P(X \in (a,b)) = \int_a^b \frac{1}{2\pi}\, d\omega$$

$$= \frac{1}{2\pi}(b-a)$$

which is proportional to the length of the interval.

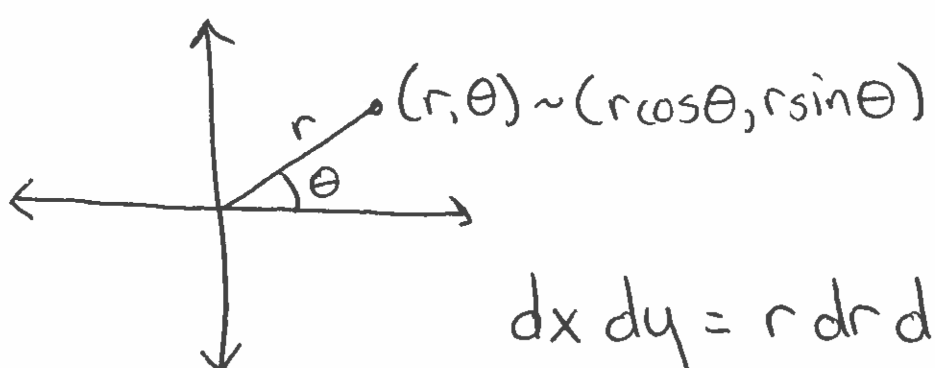
Example. ~~We throw a dart at a unit circle and let X be the position at which the dart lands.~~

Example. We raise a circular umbrella during a rainstorm and let X be the position at which the next raindrop hits the umbrella.

$$\Omega = \text{the unit circle}$$

$$\mu(\omega) = \frac{1}{\pi}$$

We recall our formulas for polar coordinates:



$$dx\,dy = r\,dr\,d\theta$$

Now we can compute

$$P(X \in \Omega) = \int_{\Omega} \frac{1}{\pi}\, dx\, dy$$

$$= \int_0^{2\pi} \int_0^1 \frac{1}{\pi} r\, dr\, d\theta$$

$$= \frac{1}{\pi} \int_0^{2\pi} \int_0^1 r\, dr\, d\theta$$

$$= \frac{1}{\pi} \int_0^{2\pi} \frac{r^2}{2} \Big|_{r=0}^{1} d\theta$$

$$= \frac{1}{\pi} \int_0^{2\pi} \frac{1}{2}\, d\theta = \frac{2\pi}{2\pi} = 1.$$

We can now compute the probability of the event

$$E_R = \left\{ \begin{array}{l} X \text{ is within } R \text{ of the} \\ \text{center of the target} \end{array} \right.$$

We compute

$$P(X \in E_R) = \int_0^{2\pi} \int_0^R \frac{1}{\pi} r \, dr \, d\theta$$

$$= 2 \int_0^R r \, dr$$

$$= 2 \frac{R^2}{2} = R^2.$$

We now consider a new random variable.

Y = distance from the center of the disk of X

This variable has a new sample space

$$\Omega = [0, 1]$$

but what is the right density?

Definition. Let X be a random variable with $\Omega = \mathbb{R}$. We define the cumulative distribution function or cdf of X to be the function F_X so that

$$F_X(x) = P(X \leq x).$$

Theorem. If X is a random variable with $\Omega = \mathbb{R}$ and density function $\mu(x)$ then the cdf F_X is given by

$$F_X(x) = \int_{-\infty}^{x} \mu(t)\, dt$$

and

$$\frac{d}{dx} F_X(x) = \mu(x)$$

Now we can observe that

$$P(X \text{ is within } R \text{ of center}) = R^2$$

$$= P(\Upsilon \leq R)$$

$$= F_{\Upsilon}(R)$$

the cdf of Υ. Thus, by our theorem

$$\mu_{\Upsilon}(R) = \frac{d}{dR} F_{\Upsilon}(R) = \frac{d}{dR} R^2 = 2R.$$

and the pdf of Υ is $2R$.

Check. We compute

$$\int_0^1 \mu_{\Upsilon}(R)\, dR = \int_0^1 2R\, dR = 1.$$

so $P(\Omega_{\Upsilon}) = 1$, as required.

Example. X is chosen uniformly from $[0,1]$, and $Y = X^2$. What is the cdf of Y? What is the pdf of Y?

Example. X_1 and X_2 are chosen uniformly from $[0,1]$ and $Y = X_1 + X_2$. What is the cdf of Y? What is the pdf of Y?

Example. X has the exponential density function $$p(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

We call $1/\lambda$ the mtbf of X.