Random Spanning Trees and the Matrix Tree Theorem.

We are now going to prove some amazing connections between eigenvalues and spanning trees.

We start with some determinant facts.

Leibniz formula. For an $n \times n$ matrix A,

$$\det A = \sum_{\pi \in S_n} \left(\operatorname{sgn}(\pi) \prod_{i=1}^{n} A_{i,\pi(i)} \right)$$

where S_n is the group of permutations of n letters.

Definition. The sign of a permutation is defined by $\operatorname{sgn}(\pi) = (-1)^{\#\text{of 2-cycles}}$.

Fact. Although the decomposition of π into 2-cycles (or transpositions) is not unique, the parity of the # of 2-cycles is the same in all decomposition of π.

Product formula: $\det(AB) = \det A \det B$.

Invariance under now operations.

For every c, if $A = \begin{pmatrix} \uparrow & & \uparrow \\ \vec{a}_1 & \cdots & \vec{a}_n \\ \downarrow & & \downarrow \end{pmatrix}$, then

$$\det\begin{pmatrix} \uparrow & & \uparrow \\ \vec{a}_1 & \cdots & \vec{a}_n \\ \downarrow & & \downarrow \end{pmatrix} = \det\begin{pmatrix} \uparrow & & \uparrow & \uparrow \\ \vec{a}_1 & \cdots & \vec{a}_{n-1} & \vec{a}_n + c\vec{a}_1 \\ \downarrow & & \downarrow & \downarrow \end{pmatrix}$$

We will also use the fact that $\det(A)$ is linear in every column of A.

Determinant as volume. If $P(A)$ is the polytope whose corners are $\left\{\sum_{i\in S}\vec{a}_i \;\middle|\; S\subset\{1,\dots,n\}\right\}$ (the parallelepiped with axes $\vec{a}_1,\dots,\vec{a}_n$.

Now let $\Pi_{\vec{a}_1}$ be the projection

$$\Pi_{\vec{a}_1}(\vec{v}) = \vec{v} - \frac{\langle\vec{v},\vec{a}_1\rangle}{\langle\vec{a}_1,\vec{a}_1\rangle}\vec{a}_1$$

onto the ~~plan~~ hyperspace normal to $\vec{a}_1$. We Know that

$$\det(\vec{a}_1,\dots,\vec{a}_n) = \det(\vec{a}_1,\Pi_{\vec{a}_1}\vec{a}_2,\dots,\Pi_{\vec{a}_1}\vec{a}_n)$$

$$= \|\vec{a}_1\|\cdot \text{volume of } P(\cancel{a_2}\,\Pi_{\vec{a}_1}\vec{a}_2,\dots,\Pi_{\vec{a}_1}\vec{a}_n)$$

but what is this volume?

~~*~~ We know that

$$\det(xI - A) = \sum_{k=0}^{n} x^{n-k} (-1)^k \sigma_k(A)$$

where $\sigma_k(A)$ is the k-th elementary symmetric polynomial in the eigenvalues of A.

$$\sigma_k(A) := \sum_{\substack{S = \{1,\dots,n\} \\ |S| = k}} \prod_{i \in S} \lambda_i$$

$$\sigma_1(A) = \lambda_1 + \dots + \lambda_n = \operatorname{tr} A$$

$$\sigma_0(A) = 1 \quad \text{(by convention)}$$

$$\vdots$$

$$\sigma_n(A) = \lambda_1 \cdots \lambda_n = \det A.$$

Notice that if we change A to a similar matrix XAX^{-1}, this doesn't change the eigenvalues. (old homework) so it doesn't change $\sigma_1, \dots, \sigma_n$ either.

We also proved that

$$\sigma_K(A) = \sum_{|S|=K} \det(A(S,S))$$

= sum of determinants of principal $K \times K$ minors.

With all this in hand,

$$\mathrm{vol}(\Pi_{\vec{a}_1}\vec{a}_2, \dots, \Pi_{\vec{a}_1}\vec{a}_n) = \sigma_{n-1}(\vec{0}, \Pi_{\vec{a}_1}\vec{a}_2, \dots, \Pi_{\vec{a}_1}\vec{a}_n)$$

because $\sigma_{n-1}(\vec{0}, \Pi_{\vec{a}_1}\vec{a}_2, \dots, \Pi_{\vec{a}_1}\vec{a}_n)$ has only one nonzero term, given by the product of the $n-1$ nonzero eigenvalues.

We can also compute

$$\mathrm{vol}\left(\Pi_{\vec{a}_1}\vec{a}_2, \ldots, \Pi_{\vec{a}_1}\vec{a}_n\right)^2$$

$$= \det\left(\left(\Pi_{\vec{a}_1}\vec{a}_2 \cdots \Pi_{\vec{a}_1}\vec{a}_n\right)^T \left(\Pi_{\vec{a}_1}\vec{a}_2 \cdots \Pi_{\vec{a}_1}\vec{a}_n\right)\right)$$

where the matrix inside the determinant is the <u>Gramian</u> matrix of dot products of these $n-1$ vectors and the det is called the <u>Gramian determinant</u>.

As a last fact, if A and B are symmetric matrices of rank K with the same nullspace,

$$\sigma_K(AB) = \sigma_K(A)\,\sigma_K(B)$$

Matrix Tree Theorem.

Let $G=(V,E,\omega)$ be a connected weighted graph,

$$\sigma_{n-1}(L_G) = n \sum_{\substack{\text{spanning} \\ \text{trees of } G}} \prod_{\substack{a \sim b \\ \in T}} \omega(a \sim b)$$

Proof. Suppose that G is a tree.

For each vertex $a \in G$, let $S_a = V - \{a\}$.

We know

$$\sigma_{n-1}(L_G) = \sum_{a \in V} \det(L_G(S_a, S_a))$$

Now we can always write

$$\underset{\substack{\uparrow \\ V\times E}}{L_G = U^T} W \underset{\substack{\uparrow \\ E\times V}}{U} \leftarrow \text{adjacency matrix}$$

$\cancel{B^{\#} W B^T}$ ← ~~boundary matrix~~ $\cancel{\mathbb{R}^E \to \mathbb{R}^V}$

where W is the $E \times E$ diagonal matrix of edge weights.

If we recall that also

$$L_G = \underbrace{(U^T W^{1/2})}_{B^{\cancel{T}}} \underbrace{(W^{1/2} U)}_{B^T}$$

$V \times E$ boundary transpose

boundary matrix $E \times V$

we have

~~det~~

$$L_G(S_a, S_a) = \underbrace{B(S_a, :)}_{\text{a square matrix}} B^T(:, S_a)$$

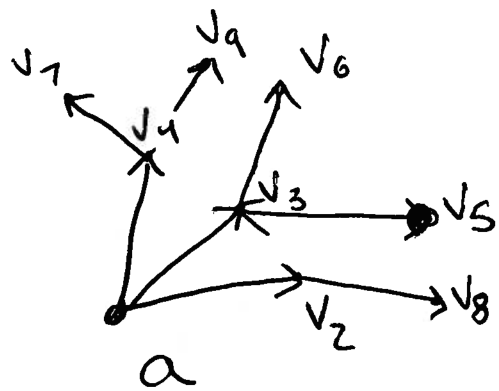
a square matrix b/c a tree has $V+1$ edges (always)

so

$$\det(L_G(S_a, S_a)) = \det(B(S_a; :))^2$$

Now we have already shown that renumbering vertices and edges permutes row and columns of B. This transformation can't change $\det B(S_a, \text{all})$.

So let's number a particular way.



Start at $a = v_1$, and number vertices by distance from a (breaking ties randomly). ~~Orient each edge by distance from a so that the head~~

Now each v_i has one edge $v_j \multimap v_i$ with $j<i$. We call this edge e_{i-1}.

Now we Know

$$B_{ij} = \begin{cases} +\sqrt{w_j} & \text{if vertex } i \text{ is the head of edge } j \\ -\sqrt{w_j} & \text{if vertex } i \text{ is the tail of edge } j \\ 0 & \text{otherwise} \end{cases}$$

so in our new numbering

$B_{i,i-1} = \pm\sqrt{w_{i-1}}$, because v_i is either head or tail of e_{i-1}

$B_{i,k} = 0$ if $k > i-1$, because the other vertex v_j incident edge e_{i-1} has $j<i$

If we now take

$$B(S_a; \text{all}) = B(\{2, \ldots, n\}, \text{all})$$

we see that

$$B(\{2, \ldots, n\}, \text{all})_{i,i} = B_{i+1,i} = \pm\sqrt{\omega_i}$$

and

$$B(\{2, \ldots, n\}, \text{all})_{i,K} = B_{i+1,K} = 0 \text{ if } K > i$$

so

$B(\{2, \ldots, n\}, \text{all})$ is lower-triangular and has determinant $\pm \prod_i \sqrt{\omega_i}$.

Thus

$$\det(B(S_a; \text{all}))^2 = \prod_i \omega_i .$$

This proves the theorem if G is a tree.

Now in general, let

$$L_G = BB^T, \text{ where } B = U^T W^{1/2}$$

We know

verts × verts

$$\sigma_{n-1}(L_G) = \sigma_{n-1}(BB^T)$$

edges × edges, because the eigenvalues are the same (up to some zeros)

$$= \sigma_{n-1}(B^TB)$$

$$= \sum_{\substack{|S| = \text{verts}-1 \\ S \subset E}} \sigma_{n-1}\big(\underbrace{B^T(S;\text{all})\,B(\text{all};S)}_{\text{principal minor}}\big)$$

$$B^TB(S,S)$$

$$= \sum_{\substack{|S| = \text{verts}-1 \\ S \subset E}} \sigma_{n-1}\big(B(\text{all},S)\,B^T(S,\text{all})\big)$$

again eigenvals are same up to zeros!

But we've done something very clever here!

$$B(\text{all}, S) = \text{verts} \times |S| \text{ matrix}$$

$$\text{where } B_{ij} = \begin{cases} +\sqrt{\omega_j} & \text{if } v_i \text{ is head of } e_j \\ -\sqrt{\omega_j} & \text{if } v_i \text{ is tail of } e_j \\ 0 & \text{, otherwise} \end{cases}$$

is the boundary operator for the subgraph of G containing only edges in S. Call this G_S. Then

$$B(\text{all}; S)\, B^T(S; \text{all}) = L_{G_S}$$

So we have shown

$$\sigma_{n-1}(L_G) = \sum_{\substack{|S| = \text{verts}-1 \\ S \subset E}} \sigma_{n-1}(L_{G_S})$$

But if G_S is connected, it is a tree, and by our previous argument, $\sigma_{n-1}(L_{G_S}) = n \prod_{e_j \in S} \omega_j$.

If G_S is disconnected, it has > 1 zero eigenvalue and $\sigma_{n-1}(L_{G_S}) = 0$.

Thus

$$\sigma_{n-1}(L_G) = n \sum_{\substack{T \text{ is a} \\ \text{spanning} \\ \text{tree of } G}} \prod_{e_j \in T} \omega_j \qquad \square .$$