Math 4250/6250 Minihomework: The Shape Operator and Spheres

This minihomework accompanies the lecture notes on "The Gauss Map and the Second Fundamental Form".

1. (20 points) Suppose that we have a surface[a] M on which the shape operator is a multiple of the identity. That is, for any $\vec{v}$ in the tangent plane T_pM, we have $S_p(\vec{v}) = k(p)\vec{v}$ where $k(p)$ is a scalar. We proved in the notes that if $k(p) = 0$, then the surface M is part of a plane. We're now going to prove that if $k(p) \neq 0$ (anywhere) then the surface M is part of a sphere.

   (1) (10 points) The first task is to prove that $k(p)$ is constant. Since $S_p(\vec{v}) = -D_{\vec{v}}n = k\vec{v}$, we can choose a C^∞ regular parametrization X of S around p and write

   $$\vec{n}_u = D_{\vec{x}_u}n = -S_p(\vec{x}_u) = -k\vec{x}_u$$

   and

   $$\vec{n}_v = D_{\vec{x}_v}n = -S_p(\vec{x}_v) = -k\vec{x}_v.$$

   Differentiate these equations to solve for the partial derivatives k_u and k_v. Then prove that k_u and k_v are zero.

[a] As usual, we're assuming that our surface is smooth– that is, it has a parametrization $X\colon U \to S$ of a neighborhood of each point $p \in S$ which is a C^∞ map.

(2) (10 points) Now consider the point $\vec{c}(u,v) = \vec{x}(u,v) + \frac{1}{k}\vec{n}(u,v)$. This point should be the center of the sphere. Prove that the location of this point doesn't depend on u and v by showing that the partials $\vec{c}_u$ and $\vec{c}_v$ are both zero.

One we've proved that $\vec{c}$ is a constant point, we know that $\|\vec{x}(u,v) - \vec{c}\| = \frac{1}{k}$ and so have proved that the surface S is a sphere centered at $\vec{c}$ with radius $\frac{1}{k}$.