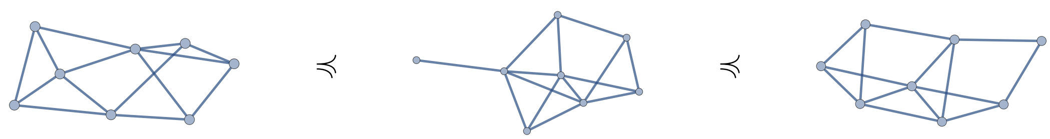
MATH/CSCI 4690/6690 : Graph comparison and the Loewner order

In this homework, we'll practice comparing graphs according the Loewner order. It's worth thinking for a moment about how weird and cool this is. It's not at all obvious that there should be any way to define even a partial order on graphs with which you could[1] make statements like



1. (10 points) Recall that we have

   Definition. *If A is a symmetric $n \times n$ matrix, we say that $A \succcurlyeq 0$ if A is a positive semidefinite matrix*[2] *If B is another $n \times n$ symmetric matrix, we say $A \succcurlyeq B \iff A - B \succcurlyeq 0$.*

   Suppose that A, B and C are symmetric $n \times n$ matrices.

   (1) (5 points) Prove that

   $$A \succcurlyeq B \text{ and } B \succcurlyeq C \implies A \succcurlyeq C.$$

[1] Actually, I just picked the graphs in this figure randomly. So I'm not claiming that they are Loewner-comparable.

[2] Remember, a matrix is P.S.D. if every eigenvalue of A is ≥ 0.

(2) (5 points) Prove that

$$A \succcurlyeq B \implies A + C \succcurlyeq B + C$$

2. (10 points) Here are two definitions of tree graphs.

Definition. *The complete binary tree* T_d *of depth* $d+1$ *is the graph whose vertices are the strings* S *of length* $0 \leq \operatorname{len} S \leq d$ *of digits* $b_1 \cdots b_{\operatorname{len}(S)}$*, where each digit* b_i *is either* 1 *or* 0*.*

If $b_1 \cdots b_k$ *is a vertex with* $k < d$ *digits, it is incident to the edges*

$$b_1 \cdots b_k \;\bullet\!\!-\!\!\bullet\; b_1 \cdots b_k 1 \quad \textit{and} \quad b_1 \cdots b_k \;\bullet\!\!-\!\!\bullet\; b_1 \cdots b_k 0$$

The string of length 0 is a special vertex called the root r *of the tree. We have*

$$r \;\bullet\!\!-\!\!\bullet\; 0 \quad \textit{and} \quad r \;\bullet\!\!-\!\!\bullet\; 1.$$

Definition. *The graph* $\mathcal{T}_d$ *has* $2^{d+1}-1$ *vertices* $1, \ldots, 2^{d+1}-1$*, where vertex* i *is incident to the edges*

$$i \;\bullet\!\!-\!\!\bullet\; 2i \quad \textit{and} \quad i \;\bullet\!\!-\!\!\bullet\; 2i+1$$

Prove that T_d is isomorphic to $\mathcal{T}_d$ by finding an explicit bijection between the vertices of T_d (binary strings) and the vertices of $\mathcal{T}_d$ (integers), and proving that your bijection of vertices induces a bijection between the edges of T_d and the edges of $\mathcal{T}_d$.

3. (10 points) Let $\vec{v}$ be any vector with $\left\langle \vec{v}, \vec{1} \right\rangle = 0$, and t be any real number. Show that

$$\|\vec{v}\| \leq \|\vec{v} + t\vec{1}\|$$

4. (10 points) The *Kirchhoff index* of a graph $\mathbf{G}$ is the sum of the reciprocals of the (nonzero) eigenvalues of the graph Laplacian:

$$K_{\mathbf{G}} = \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_{\mathbf{v}}}$$

Suppose that $\mathbf{G}$ is a connected graph, and $\mathbf{H}$ is a connected graph obtained from $\mathbf{G}$ by deleting vertices. Prove that $K_{\mathbf{H}} < K_{\mathbf{G}}$.

Hint: Think "eigenvalue interlacing".

5. (10 points) The *Kirchhoff index* of a graph $\mathbf{G}$ is the sum of the reciprocals of the (nonzero) eigenvalues of the graph Laplacian:

$$K_{\mathbf{G}} = \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_{\mathbf{v}}}$$

Suppose that $\mathbf{G}$ is a connected graph, and $\mathbf{H}$ is a connected subgraph[3] of $\mathbf{G}$. Prove that $K_{\mathbf{G}} < K_{\mathbf{H}}$.[4]

Hint: Think "Loewner ordering".

[3] Remember that a subgraph is obtained by deleting edges, but not vertices, of a graph.

[4] The ordering and the inequality are NOT misprints. You're really proving the *opposite* inequality from the previous question. Weird, right?