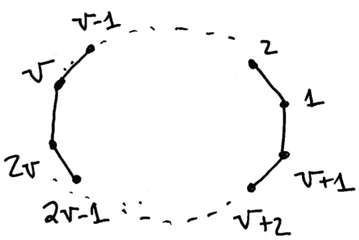
MATH/CSCI 4690/6690 : Path and Cycle Graphs

In this homework, we'll work out some computations which support our calculation of the eigenvalues and eigenvectors of the path and cycle graphs.

1. (10 points) Suppose that we renumber the vertices in the cycle graph $C_{2\mathbf{v}}$ as below:



so that the edges of $C_{2\mathbf{v}}$ are

$$1 \bullet\!\!-\!\!\bullet 2 \bullet\!\!-\!\!\bullet \cdots \bullet\!\!-\!\!\bullet \cdots \bullet\!\!-\!\!\bullet \mathbf{v}-1 \bullet\!\!-\!\!\bullet \mathbf{v} \bullet\!\!-\!\!\bullet 2\mathbf{v} \bullet\!\!-\!\!\bullet 2\mathbf{v}-1 \bullet\!\!-\!\!\bullet \cdots \bullet\!\!-\!\!\bullet \mathbf{v}+2 \bullet\!\!-\!\!\bullet \mathbf{v}+1 \bullet\!\!-\!\!\bullet 1.$$

Further, suppose that we define the path graph $P_{\mathbf{v}}$ (as usual) as the graph with vertices $1, \dots, \mathbf{v}$ and edges

$$1 \bullet\!\!-\!\!\bullet 2 \bullet\!\!-\!\!\bullet \dots \bullet\!\!-\!\!\bullet \mathbf{v}-1 \bullet\!\!-\!\!\bullet \mathbf{v}.$$

Prove that the Laplacian $L_{C_{2\mathbf{v}}}$ of the renumbered cycle graph is related to the Laplacian $L_{P_{\mathbf{v}}}$ of the path graph by

$$\begin{bmatrix} I_{\mathbf{v}} & I_{\mathbf{v}} \end{bmatrix} L_{C_{2\mathbf{v}}} \begin{bmatrix} I_{\mathbf{v}} \\ I_{\mathbf{v}} \end{bmatrix} = 2 L_{P_{\mathbf{v}}}.$$

2. (10 points) We proved in the notes that

Proposition. *The cycle graph* $C_{2\mathbf{v}}$ *with vertices* $1, \ldots, 2\mathbf{v}$ *and edges*

$$1 \bullet\!\!-\!\!\bullet 2 \bullet\!\!-\!\!\bullet \cdots \bullet\!\!-\!\!\bullet \cdots \bullet\!\!-\!\!\bullet 2\mathbf{v}-1 \bullet\!\!-\!\!\bullet 2\mathbf{v} \bullet\!\!-\!\!\bullet 1$$

has eigenvectors

$$\begin{aligned} \vec{x}_k(a) &= \cos\left(\pi k \frac{a}{\mathbf{v}}\right) \\ \vec{y}_k(a) &= \sin\left(\pi k \frac{a}{\mathbf{v}}\right) \end{aligned} \tag{$\star$}$$

for each integer k *with* $0 \le k \le \mathbf{v}$ *except for* $\vec{y}_0 = \vec{0}$ *and* $\vec{y}_{\mathbf{v}} = \vec{0}$. *Eigenvectors* $\vec{x}_k$ *and* $\vec{y}_k$ *have eigenvalue* $2 - 2\cos\left(\pi \frac{k}{\mathbf{v}}\right)$.

Prove that if we renumber the vertices of the cycle graph $C_{2\mathbf{v}}$ as in the first problem:

$$1 \bullet\!\!-\!\!\bullet 2 \bullet\!\!-\!\!\bullet \cdots \bullet\!\!-\!\!\bullet \cdots \bullet\!\!-\!\!\bullet \mathbf{v}-1 \bullet\!\!-\!\!\bullet \mathbf{v} \bullet\!\!-\!\!\bullet 2\mathbf{v} \bullet\!\!-\!\!\bullet 2\mathbf{v}-1 \bullet\!\!-\!\!\bullet \cdots \bullet\!\!-\!\!\bullet \mathbf{v}+2 \bullet\!\!-\!\!\bullet \mathbf{v}+1 \bullet\!\!-\!\!\bullet 1$$

in each eigenspace of $C_{2\mathbf{v}}$ there is exactly one eigenvector $\vec{z}_k$ with

$$\vec{z}_k(a) = \vec{z}_k(a + \mathbf{v})$$

(in the new numbering) for $a \in 1, \ldots, \mathbf{v}$.

Hint: First, observe that vertex $a + \mathbf{v}$ in the new numbering scheme is vertex $2\mathbf{v} - a$ in the original numbering scheme. Then work with ($\star$) to show that there's a unique linear combination of $\vec{x}_k$ and $\vec{y}_k$ which has the property that you want.