Math/Csci 4690/6690 : Elementary properties of the graph Laplacian

In this minihomework, we will prove some of the basic properties of the graph Laplacian. First, recall from the notes that

Definition. *The* boundary map *for a graph* $\mathbf{G}$ *is the* $\mathbf{v} \times \mathbf{e}$ *matrix defined by*

$$\partial(e_i) = \mathrm{head}(e_i) - \mathrm{tail}(e_i)$$

The graph Laplacian $\mathrm{L}_{\mathbf{G}}$ *for a graph* $\mathbf{G}$ *is the symmetric* $\mathbf{v} \times \mathbf{v}$ *matrix defined by*

$$\mathrm{L}_{\mathbf{G}} := \partial\partial^T.$$

We proved in the notes that for any vector $v \in \mathbb{R}^{\mathbf{v}}$, we have

$$\mathrm{Q}_{\mathrm{L}_{\mathbf{G}}}(v) = \langle v, \mathrm{L}_{\mathbf{G}}\, v\rangle = \sum_{i=1}^{\mathbf{e}} \left(v_{\mathrm{head}(e_i)} - v_{\mathrm{tail}(e_i)}\right)^2.$$

1. (10 points) Suppose that $\mathbf{G}$ and $\mathbf{G}'$ have the same vertices and edges $v_1, \dots, v_{\mathbf{v}} = v'_1, \dots, v'_{\mathbf{v}'}$ and $e_1, \dots, e_{\mathbf{e}} = e'_1, \dots, e'_{\mathbf{e}'}$, but the orientations of the edges may not agree.

   Prove that $\mathrm{L}_{\mathbf{G}} = \mathrm{L}_{\mathbf{G}'}$.

2. (10 points) Recall that

Definition. *The degree matrix $D_{\mathbf{G}}$ is the diagonal matrix whose entries are the degrees of the vertices $v_1, \ldots, v_{\mathbf{v}}$, and the adjacency matrix $M_{\mathbf{G}}$ of $\mathbf{G}$ is the symmetric matrix defined by*

$$(M_{\mathbf{G}})_{ij} = \begin{cases} 1, & \textit{if } v_i \textit{ and } v_j \textit{ are joined by an edge} \\ 0, & \textit{if not.} \end{cases}$$

Prove that $\mathrm{L}_{\mathbf{G}} = D_{\mathbf{G}} - M_{\mathbf{G}}$.

3. (10 points) Let $\lambda_1 \leq \cdots \leq \lambda_n$ be the eigenvalues of the symmetric matrix $\mathrm{L}_{\mathbf{G}}$.

(1) (5 points) Prove that $\lambda_1 = 0$ by finding a vector $v \in \mathbb{R}^{\mathbf{V}}$ so that $\mathrm{L}_{\mathbf{G}}\, v = 0$.

(2) (5 points)

Definition. *A set $\mathcal{S}$ of vertices of* $\mathbf{G}$ *is called a* component *of* $\mathbf{G}$ *if there are no edges joining vertices in $\mathcal{S}$ to vertices outside $\mathcal{S}$. The largest set of disjoint components $\mathcal{S}_1, \ldots, \mathcal{S}_d$ of* $\mathbf{G}$ *are called the* connected components *of* $\mathbf{G}$.

Prove that if $\mathbf{G}$ has d connected components, then $\lambda_1 = \lambda_2 = \ldots = \lambda_d = 0$.

Hint: The $\mathcal{S}_i$ really partition $\mathbf{G}$ into d separate graphs $\mathbf{G}_1, \ldots, \mathbf{G}_d$.