MATH/CSCI 4690/6690 : Graph coloring and spectral theory

In this homework, we'll practice using our theorems about graph colorings and the spectrum of the adjacency matrix. To review, these are:

Definition. *The adjacency matrix of a graph* $\mathbf{G}$*, denoted* $\mathrm{M}_{\mathbf{G}}$*, is the symmetric matrix where*

$$(\mathrm{M}_{\mathbf{G}})_{ij} = \begin{cases} 1, & \text{if } v_i \leftrightarrow v_j \text{ is an edge of } \mathbf{G} \\ 0 & \text{if not.} \end{cases}$$

We denote the eigenvalues of $\mathrm{M}_{\mathbf{G}}$ *by* $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{\mathbf{v}}$.

Theorem. *If* S *is any subgraph of* $\mathbf{G}$*,* $\mathrm{d}_{ave}(S)$ *is the average degree of vertices in the subgraph, and* $\mathrm{d}_{\max}$ *is the maximum degree of any vertex in* $\mathbf{G}$*, then*

$$\mathrm{d}_{ave}(S) \leq \mu_1 \leq \mathrm{d}_{\max}$$

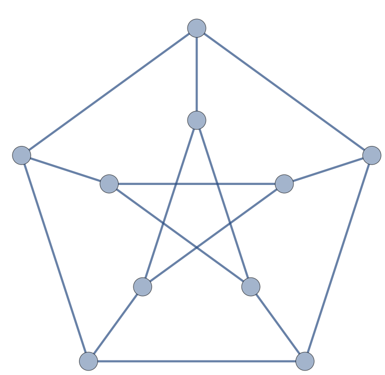
Theorem. *If* $\mathbf{G}$ *is connected and* $\mu_1 = \mathrm{d}_{\max}$*, then* $\mathbf{G}$ *is* $\mathrm{d}_{\max}$*-regular.*[1] *If* $\mathbf{G}$ *is* d*-regular, then* $\mu_1 = d$.

Definition. *A* coloring *of a graph is an assignment of colors to vertices so that every edge joins vertices of different colors. A graph is* k-colorable *if a coloring exists with* k *colors. The* chromatic number $\chi(\mathbf{G})$ *of a graph is the smallest* k *for which* $\mathbf{G}$ *is* k*-colorable.*

Theorem. *For any graph* $\mathbf{G}$*,* $\chi(\mathbf{G}) \leq \lfloor \mu_1 \rfloor + 1$.

Theorem. *If* $\mu_1 = -\mu_{\mathbf{v}}$ *then* $\mathbf{G}$ *is 2-colorable.*[2] *If* $\mathbf{G}$ *is 2-colorable, then the eigenvalues* $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{\mathbf{v}}$ *are symmetric around 0.*

1. (20 points) Consider the graph

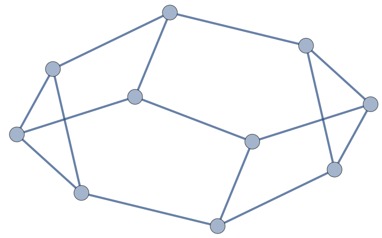


[1] Every vertex has degree d.
[2] A graph which is 2-colorable is said to be *bipartite*.

(1) (10 points) Establish an upper bound on $\chi(\mathbf{G})$ by finding a coloring of $\mathbf{G}$ with as few colors as possible.

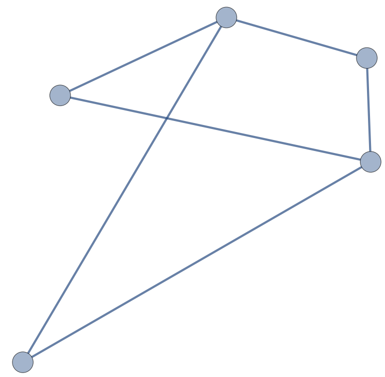
(2) (10 points) Prove that your upper bound is actually equal to $\chi(\mathbf{G})$ by using our theorems to show that no coloring with fewer colors exists. (If you need eigenvalues of $\mathrm{M}_{\mathbf{G}}$, it's expected that you'll use a computer to find them. This is an acceptable proof technique as long as you include screenshots.)

2. (10 points) Consider the graph



Find the best bounds on $\chi(\mathbf{G})$ that you can by explicitly finding colorings and using our theorems above. Can you compute $\chi(\mathbf{G})$ exactly?

3. (10 points) Consider the graph



Find as many eigenvalues of $\mathrm{M}_{\mathbf{G}}$ as you can *without* using a computer; prove that each number you give is actually an eigenvalue of $\mathrm{M}_{\mathbf{G}}$ either by substituting it into the characteristic polynomial and verifying that it is a root, or by giving the corresponding eigenvector.