Math 4250 Minihomework: Self-Adjoint Operators

This minihomework accompanies the lecture notes on "The Second Fundamental Form (Part II)". In our discussion of the shape operator and the second fundamental form, we introduced an interesting new idea. First, we recall

Definition. *Suppose we have a d-dimensional vector space V and a map $A\colon\ V \to V$. We say that A is a* linear *map if for each $\vec{v}, \vec{w} \in V$ and $\lambda, \mu \in \mathbb{R}$ we have*

$$A(\lambda\vec{v} + \mu\vec{w}) = \lambda A(\vec{v}) + \mu A(\vec{w})$$

We so often identify a linear map between d-dimensional vector spaces with a $d \times d$ matrix that we forget that there's a missing ingredient required to associate a matrix with a linear map:

Definition. *If $\vec{v}_1, \dots, \vec{v}_d$ is a basis for the d-dimensional vector space V, then any linear map $A\colon\ V \to V$ can be written (uniquely) in terms of a $d \times d$ matrix $[A] = [a_{ij}]$ whose j-th column is defined by the equation*

$$A\vec{v}_i = a_{i1}\vec{v}_1 + \cdots + a_{id}\vec{v}_d.$$

In this case, if $\vec{x} = x_1\vec{v}_1 + \cdots + x_d\vec{v}_d$ and $A\vec{x} = b_1\vec{v}_1 + \cdots + b_d\vec{v}_d$, we can write

$$[A]\begin{bmatrix} x_1 \\ \vdots \\ x_d \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1d} \\ \vdots & \ddots & \vdots \\ a_{d1} & \cdots & a_{dd} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_d \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_d \end{bmatrix}$$

as we are used to doing automatically in linear algebra. The important point here is that the matrix $[A]$ is not determined only by linear map A– *it's determined by the choice of basis as well.* The same linear map will have different matrix representations in different bases for V.

We'd like to understand when a linear map A will be represented by a symmetric matrix $[A]$. To this end, we define:

Definition. *Suppose we have a d-dimensional vector space V, a linear map $A\colon\ V \to V$, and a quadratic form $Q\colon\ V \times V \to \mathbb{R}$. We say that A is* self-adjoint with respect to Q *if for each $\vec{v}$, $\vec{w} \in V$, we have*

$$Q(A\vec{v}, \vec{w}) = Q(\vec{v}, A\vec{w}).$$

We are now going to explore the connection between self-adjoint linear maps[a] and symmetric matrices through a few exercises.

[a]These are also called "self-adjoint operators".

1. (10 points) Suppose we choose a basis $\vec{v}_1, \vec{v}_2$ for the 2-dimensional vector space V and have a linear map $A\colon V \to V$ represented by the matrix

$$[A] = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$$

and a quadratic form $Q\colon V \times V \to \mathbb{R}$ given by the matrix

$$[Q] = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

Is A self-adjoint with respect to Q? Prove it.

Solution:

2. (20 points) Suppose A is a linear map $V \to V$ and Q is a quadratic form on V and A is self-adjoint with respect to Q. Further, suppose that $\vec{v}_1, \ldots, \vec{v}_d$ is any basis for V so that $Q(\vec{v}_i, \vec{v}_j) = 0$ when $i \neq j$.[b]

   Prove that the matrix $[A]$ representing A in the basis $\vec{v}_1, \ldots, \vec{v}_d$ is symmetric.

   Solution:

[b] This means that the matrix $[Q]$ representing Q in the basis $\vec{v}_1, \ldots, \vec{v}_d$ is diagonal.

Solution: