A General Cross-Helicity Formula

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Abstract. In this paper, we consider the mutual helicity of a pair of divergence–free vector fields defined on, and tangent to the boundaries of two disjoint domains in 3-space. Generalizing previous results of Moffatt, and Laurence and Avellaneda, we derive a simple formula for this helicity in terms of the linking numbers of curves on the boundary of the two domains, and fluxes of the vector fields across corresponding spanning surfaces inside the domains.

1. Introduction

Suppose we have disjoint domains Ω and Ω' in $\mathbf{R}^3$ and divergence–free vector fields V and V' defined on Ω and Ω' and tangent to their boundaries. The cross-helicity of these fields is the integral

$$\mathrm{H}(V, V') = \frac{1}{4\pi} \int_{\Omega \times \Omega'} V(p) \times V'(q) \cdot \frac{p-q}{|p-q|^3} \, \mathrm{dVol}_p \, \mathrm{dVol}_q \, .$$

In 1969, H.K. Moffatt [8] showed that for a pair of linked solid tori Ω and Ω' of small cross-section,

$$\mathrm{H}(V, V') = \mathrm{Lk}(\Omega, \Omega') \, \mathrm{Flux}(V) \, \mathrm{Flux}(V').$$

This formula was revisited by Laurence and Avellaneda [7] in 1992, who extended the result to a pair of arbitrary solid tori using a clever application of Arnold's idea [1] of the "asymptotic linking number." (See [2] for another version of this formula.) These results all prescribe the topology of Ω and Ω'.

Our goal is to derive the corresponding formula for domains of arbitrary topology in 3-space. We will be able to give a general formula for the cross-helicity $H(V, V')$ in terms of the linking numbers of curves constructed from Ω and Ω' and the fluxes of V and V' over corresponding spanning surfaces in those domains. This paper summarizes Chapter 8 of my thesis [4]. I am grateful for the generous support of the NSF Postdoctoral Research Fellowship Program.

2. Choosing Curves and Spanning Surfaces

We begin by choosing a basis for the first homology of Ω, represented by curves $C_1, \ldots, C_n$ on the boundary of Ω. These curves are dual to a set of spanning surfaces for Ω, denoted $S_1, \ldots, S_n$. Poincaré duality guarantees that these surfaces represent a basis for the second homology of Ω, relative to the boundary of Ω.

In practice, we choose the S_i by checking that the intersection number of C_i with S_j is δ_{ij}. We can double-check our work by verifying that n is equal to the total genus of the boundary of Ω.

3. Statement of the theorem

We can now state the theorem.

Theorem 1. *For any choice of C_i and S_i for Ω, and C'_j and S'_j for Ω' as above, the cross-helicity of a pair of divergence-free vector fields V and V', tangent to the smooth boundaries of two compact domains Ω and Ω' in 3-space, is given by*

$$\mathrm{H}(V,V') = \sum_{i,j} \mathrm{Lk}(C_i, C'_j)\,\mathrm{Flux}(V,S_i)\,\mathrm{Flux}(V',S'_j).$$

Figure 1 illustrates the method when Ω and Ω' are solid tori.

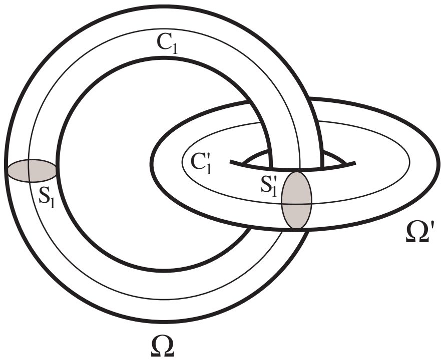

Figure 1. Both domains are solid tori.

In this case, $\mathrm{Lk}(C_1, C'_1) = 1$, and so

$$\mathrm{H}(V,V') = \mathrm{Flux}(V,S_1)\,\mathrm{Flux}(V',S'_1).$$

Now, to see the power of this method, we'll try a more complicated domain, shown in Figure 2, and get a more complicated formula.

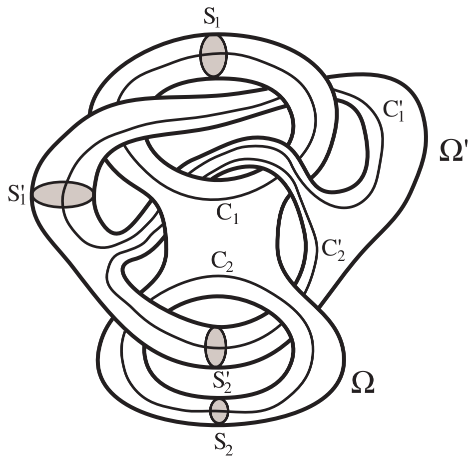

Figure 2: A more complicated pair of domains.

Here, we have

$$\begin{array}{ll} \mathrm{Lk}(C_1, C'_1) = 2. & \mathrm{Lk}(C_1, C'_2) = 1. \\ \mathrm{Lk}(C_2, C'_1) = 0. & \mathrm{Lk}(C_2, C'_2) = -1. \end{array}$$

so the helicity is given by

$$\begin{aligned}
\mathrm{H}(V, V') &= 2\,\mathrm{Flux}(V, S_1)\,\mathrm{Flux}(V', S'_1) \\
&+ \mathrm{Flux}(V, S_1)\,\mathrm{Flux}(V', S'_2) \\
&- \mathrm{Flux}(V, S_2)\,\mathrm{Flux}(V', S'_2).
\end{aligned}$$

We will start by recalling some standard results about the topology of domains in 3-space, and by recalling the standard correspondance between vector fields and cohomology. These topics are covered in detail in the books [9] and [6].

4. The topology, homology, and cohomology of domains in 3-space.

Take a compact domain Ω in 3-space, with boundary $\partial\Omega$, such as a ball, or a solid torus. The boundary is a surface, or a collection of surfaces, and each surface is topologically equivalent to a torus with a number of holes. The number of holes is called the *genus* of the surface. For example, each domain in the example above is bounded by a surface of genus two.

We need to study the homology and cohomology of these domains, with real coefficients. In this world, the first homology of Ω is a real vector space of equivalence classes of closed, oriented curves in Ω, where two curves are considered equivalent if they bound a common surface inside the domain, and the sign of a curve depends on its orientation. The first homology of Ω is denoted $H_1(\Omega)$.

The second homology of Ω, relative to the boundary of Ω, is a vector space of equivalence classes of "spanning surfaces" for Ω, whose boundaries lie on the boundary of Ω. Two such surfaces are considered equivalent if they, together with any portion of $\partial\Omega$, bound a common volume inside Ω. Again, the sign of a surface depends on its orientation. The second relative homology of Ω is denoted $H_2(\Omega, \partial\Omega)$.

The cohomology spaces of Ω are the dual vector spaces to the homology vector spaces. The cohomology spaces of Ω that we are interested in are the first cohomology, denoted $H^1(\Omega)$ and the second relative cohomology $H^2(\Omega, \partial\Omega)$. There is an operation, called the cup product, which takes a class in $H^1(\Omega)$, together with a class in $H^2(\Omega, \partial\Omega)$, and (effectively) returns a real number.

If we have a curve C and a spanning surface S, dual to $C^* \in H^1(\Omega)$ and to $S^* \in H^2(\Omega, \partial\Omega)$, then the cup product $C^* \cup S^*$ is equal to the (signed) intersection number of C and S, or the linking number of C with the boundary of S.

A beautiful result called Poincaré duality tells us that the vector spaces $H_1(\Omega)$ and $H_2(\Omega, \partial\Omega)$ are isomorphic. Further, if we take a basis for $H_1(\Omega)$ represented by curves $C_1, \ldots, C_n$, and the corresponding dual basis for $H_2(\Omega, \partial\Omega)$, represented by surfaces $S_1, \ldots, S_n$, the intersection number of C_i and S_j will be δ_{ij}.

5. Vector fields and cohomology

There is also a canonical correspondence between vector fields and cohomology classes. A vector field V on Ω defines a linear functional on closed curves inside Ω by taking the circulation of V around the curve:

$$V(C) = \int_C V \cdot \mathrm{ds}\,.$$

If V is curl-free on Ω, then V defines a linear functional on homology classes of curves, for if C and C' together bound S, then by Stokes' Theorem we have

$$V(C) - V(C') = \int_C V \cdot \mathrm{ds} - \int_{C'} V \cdot \mathrm{ds} = \int_S \nabla \times V \cdot n \,\mathrm{dArea} = 0.$$

Thus a curl-free vector field corresponds to a class in $H^1(\Omega)$.

A vector field V on Ω defines a linear functional on spanning surfaces inside Ω by taking the flux of V across the surface:

$$V(S) = \int_S V \cdot n \,\mathrm{dArea}\,.$$

If V is divergence-free and tangent to the boundary of Ω, then V defines a linear functional on homology classes of spanning surfaces, for if S and S', together with some portion B of $\partial\Omega$, bound a volume D inside Ω, then by the Divergence Theorem, we have

$$\int_S V \cdot n \,\mathrm{dArea} - \int_{S'} V \cdot n \,\mathrm{dArea} + \int_B V \cdot n \,\mathrm{dArea} = \int_D \nabla \cdot V \,\mathrm{dVol} = 0.$$

But V is tangent to $\partial\Omega$, so this proves that

$$V(S) - V(S') = \int_S V \cdot n \,\mathrm{dArea} - \int_{S'} V \cdot n \,\mathrm{dArea} = 0.$$

So a divergence-free vector field, tangent to $\partial\Omega$ corresponds to a class in $H^2(\Omega, \partial\Omega)$.

In the world of vector fields, the cup product corresponds to the L^2 inner product:

$$V \cup W = \langle V, W \rangle = \int_\Omega V \cdot W \,\mathrm{dVol}\,.$$

We have now introduced all the technology we'll need to prove our formula.

6. Rewriting the Helicity Integral

It's standard to rewrite the helicity integral $\mathrm{H}(V, V')$ as:

$$\begin{aligned}
\mathrm{H}(V, V') &= \frac{1}{4\pi} \int_{\Omega \times \Omega'} V(p) \times V'(q) \cdot \frac{p - q}{|p - q|^3} \,\mathrm{dVol}_p \,\mathrm{dVol}_q \\
&= \int_\Omega V(p) \cdot \left[\frac{1}{4\pi} \int_{\Omega'} V'(q) \times \frac{p - q}{|p - q|^3} \,\mathrm{dVol}_q \right] \mathrm{dVol}_p \\
&= \langle V, \mathrm{BS}(V') \rangle\,,
\end{aligned}$$

where, if we view V' as an electrical current on Ω', $\mathrm{BS}(V')$ is the resulting magnetic field in 3-space. It's a well known fact from physics that the curl of the magnetic field $\mathrm{BS}(V')$ is equal to the current flow V' on Ω, and that the curl $\mathrm{BS}(V')$ vanishes elsewhere. In addition, the the magnetic field $\mathrm{BS}(V')$ is everywhere divergence-free.

7. Reduction to a product of harmonic knots

We now set out to prove our result. We start with the Hodge Decomposition Theorem [5].

Theorem 2. *We have a direct sum decomposition of* $\mathrm{VF}(\Omega)$ *into five mutually orthogonal subspaces,*

$$\mathrm{VF}(\Omega) = \mathrm{FK} \oplus \mathrm{HK} \oplus \mathrm{CG} \oplus \mathrm{HG} \oplus \mathrm{GG},$$

with

$$\begin{aligned}
\mathrm{Ker\,Curl} &= \phantom{\mathrm{FK} \oplus{}} \mathrm{HK} \oplus \mathrm{CG} \oplus \mathrm{HG} \oplus \mathrm{GG} \\
\mathrm{Image\,Grad} &= \phantom{\mathrm{FK} \oplus \mathrm{HK} \oplus{}} \mathrm{CG} \oplus \mathrm{HG} \oplus \mathrm{GG} \\
\mathrm{Image\,Curl} &= \mathrm{FK} \oplus \mathrm{HK} \oplus \mathrm{CG} \\
\mathrm{Ker\,Div} &= \mathrm{FK} \oplus \mathrm{HK} \oplus \mathrm{CG} \oplus \mathrm{HG}
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{FK} &= \{\nabla \cdot V = 0,\ V \cdot n = 0, \text{ all interior fluxes } = 0\}, \\
\mathrm{HK} &= \{\nabla \cdot V = 0, \nabla \times V = 0, V \cdot n = 0\}, \\
\mathrm{CG} &= \{V = \nabla\varphi, \nabla \cdot V = 0, \text{all boundary fluxes} = 0\}, \\
\mathrm{HG} &= \{V = \nabla\varphi, \nabla \cdot V = 0, \varphi \text{ loc. constant on } \partial\Omega\}, \\
\mathrm{GG} &= \{V = \nabla\varphi,\ \varphi|_{\partial\Omega} = 0\},
\end{aligned}$$

and furthermore,

$$\begin{aligned}
\mathrm{HK} &\cong H_1(\Omega; \mathbf{R}) \cong H_2(\Omega, \partial\Omega; \mathbf{R}) \\
&\cong \mathbf{R}^{\text{genus of } \partial\Omega}. \\
\mathrm{HG} &\cong H_2(\Omega; \mathbf{R}) \cong H_1(\Omega, \partial\Omega; \mathbf{R}) \\
&\cong \mathbf{R}^{(\#\text{ components of } \partial\Omega) - (\#\text{ components of } \Omega)}.
\end{aligned}$$

We refer to FK as “fluxless knots”, HK as “harmonic knots”, CG as “curly gradients”, HG as “harmonic gradients”, and GG as “grounded gradients”.

Since V is divergence-free and tangent to $\partial\Omega$, $V \in \mathrm{FK} \oplus \mathrm{HK}$. Since $\mathrm{BS}(V')$ is divergence and curl free on Ω, $\mathrm{BS}(V') \in \mathrm{HK} \oplus \mathrm{CG} \oplus \mathrm{HG}$. This means that the inner product of V and $\mathrm{BS}(V')$ is completely determined by the inner product of the components of these fields in the subspace HK.

8. Projection of V to HK

We just observed that $V = F + H$, where $F \in \mathrm{FK}$ and $H \in \mathrm{HK}$. Let's write down a lemma.

Lemma 1. *Given any* S_i *in our basis for* $H_2(\Omega, \partial\Omega)$*, for* V *and* H *as above, we have*

$$\mathrm{Flux}(V, S_i) = \mathrm{Flux}(H, S_i).$$

Since $V = F + H$, and F is fluxless by construction, this is easy.

9. Projection of $BS(V')$ to HK

We also know that $\mathrm{BS}(V') = H + \nabla f$, where $H \in \mathrm{HK}$. We have a corresponding lemma.

Lemma 2. *Given a closed curve C_i in our basis for $H_1(\Omega)$, if*

$$\mathrm{BS}(V') = H + \nabla f,$$

where $H \in \mathrm{HK}$, then

$$\mathrm{Circ}(\mathrm{BS}(V'), C_i) = \mathrm{Circ}(H, C_i).$$

where $\mathrm{Circ}(V, C)$ denotes the circulation of the vector field V around the curve C.

Since ∇f is a gradient, $\mathrm{Circ}(\nabla f, C_i) = 0$.

10. Inner Products of Harmonic Knots

We now find a general formula for the inner product of two vector fields in $\mathrm{HK}(\Omega)$. To do so, we'll rederive Proposition 5, from [5], which was inspired by a corresponding result of Blank, Friedrichs, and Grad [3].

Proposition 1. *Suppose C_i and S_i represent bases for $H_1(\Omega)$ and $H_2(\Omega, \partial\Omega)$, chosen as above. Then, for any pair of harmonic knots V and W, we have*

$$\langle V, W \rangle = \sum_i \mathrm{Circ}(V, C_i)\, \mathrm{Flux}(W, S_i).$$

When we introduced the correspondence between vector fields and cohomology classes, we mentioned that curl-free vector fields were equivalent to one-dimensional cohomology classes, and divergence-free vector fields tangent to $\partial\Omega$ were equivalent to two-dimensional relative cohomology classes. Members of HK are *both* curl-free *and* divergence-free and tangent to $\partial\Omega$. Thus, we are free to view V as a one-dimensional class, and W as a two-dimensional relative class.

Now the C_i and S_i represent bases for $H_1(\Omega)$ and $H_2(\Omega, \partial\Omega)$, so their duals C_i^* and S_i^* represent bases for $H^1(\Omega)$ and $H^2(\Omega, \partial\Omega)$. Writing $V = \sum c_i C_i^*$ and $W = \sum s_i S_i^*$ in these dual bases, it is clear that

$$c_i = \mathrm{Circ}(V, C_i), \text{ and } s_i = \mathrm{Flux}(W, S_i).$$

The inner product $\langle V, W \rangle$ is equal to the cup product $V \cup W$. But the cup product $C_i^* \cup S_j^*$ is just the intersection number of C_i and S_j, and since we've chosen the S_j to be the Poincaré duals of the C_i, this intersection number is δ_{ij}. Thus

$$\begin{aligned}
\langle V, W \rangle = V \cup W &= \left(\sum_i c_i C_i^* \right) \cup \left(\sum_j s_j S_j^* \right) \\
&= \sum_{ij} c_i s_j \left(C_i^* \cup S_j^* \right) = \sum_{ij} c_i s_j \delta_{ij} \\
&= \sum_i c_i s_i = \sum_i \mathrm{Circ}(V, C_i)\, \mathrm{Flux}(W, S_i).
\end{aligned}$$

11. Combining results obtained so far

Combining Lemma 1 and Lemma 2 with Proposition 1, we have proved that

$$\langle V, \mathrm{BS}(V')\rangle = \sum_i \mathrm{Flux}(V, S_i)\,\mathrm{Circ}(\mathrm{BS}(V'), C_i).$$

Our goal now must be to prove that

$$\mathrm{Circ}(\mathrm{BS}(V'), C_i) = \sum_j \mathrm{Lk}(C_i, C_j')\,\mathrm{Flux}(V', S_j').$$

We will show in Proposition 2 that every homology class of curves in Ω is represented by the boundary of some spanning surface E outside Ω. Let C_i reside in the same homology class as ∂E_i. Then by Stokes' theorem,

$$\mathrm{Circ}(\mathrm{BS}(V'), C_i) = \mathrm{Circ}(\mathrm{BS}(V'), \partial E_i) = \mathrm{Flux}(\nabla \times \mathrm{BS}(V'), E_i).$$

But $\nabla \times \mathrm{BS}(V')$ is equal to V' inside Ω', and is zero elsewhere. We've proved that

$$\mathrm{Circ}(\mathrm{BS}(V'), C_i) = \mathrm{Flux}(V', E_i).$$

It remains to compute the fluxes of V' over the E_i.

12. The flux of V' over E_i

Suppose that we've chosen curves C_i' and Poincaré dual surfaces S_i' representing bases for the homology of Ω'. The intersection of E_i with Ω' is certainly a spanning surface inside Ω', so it must be equivalent to a linear combination of the surfaces S_i'. That is,

$$E_i = \sum_j e_{ij} S_j'.$$

To compute the coefficients e_{ij}, compute the intersection number of C_j' with E_i; since the S_j' are Poincaré duals of the C_j', this intersection number is equal to e_{ij}. Since C_i is the boundary of E_i, this proves that

$$E_i = \sum_j \mathrm{Int}(C_j', E_i) S_j' = \sum_j \mathrm{Lk}(C_j', \partial E_i) S_j' = \sum_j \mathrm{Lk}(C_j', C_i) S_j'.$$

In particular, the flux of V' over E_i is given by

$$\mathrm{Flux}(V', E_i) = \sum_j \mathrm{Lk}(C_i, C_j')\,\mathrm{Flux}(V', S_j'),$$

which completes the proof of the Theorem.

13. The Connecting Homomorphism

In Section 11, we claimed that every homology class of curves in $H_1(\Omega)$ was represented by the boundary of some spanning surface in $\mathbf{R}^3 - \Omega$. We now prove this result, drawing on some of the standard tools of algebraic topology:

Proposition 2. *Consider the boundary map δ, which replaces a spanning surface in $\mathbf{R}^3 - \Omega$ with it's boundary curve on $\partial\Omega$. The induced map from $H_2(\mathbf{R}^3 - \Omega, \partial\Omega)$ to $H_1(\partial\Omega)$, when composed with the map induced by inclusion, $H_1(\partial\Omega) \to H_1(\Omega)$, yields an isomorphism*

$$H_2(\mathbf{R}^3 - \Omega, \partial\Omega) \cong H_1(\Omega).$$

Our first tool is the Mayer-Vietoris exact sequence for homology. In our case, this sequence becomes

$$\to H_2(\mathbf{R}^3) \to H_1(\partial\Omega) \to H_1(\Omega) \oplus H_1(\mathbf{R}^3 - \Omega) \to H_1(\mathbf{R}^3) \to$$

Since $H_2(\mathbf{R}^3) = H_1(\mathbf{R}^3) = 0$, the middle arrow must be an isomorphism between $H_1(\partial\Omega)$ and $H_1(\Omega) \oplus H_1(\mathbf{R}^3 - \Omega)$. This isomorphism is induced by the inclusion maps from $\partial\Omega$ into Ω and into $\mathbf{R}^3 - \Omega$.

We can use Mayer-Vietoris to derive a similar isomorphism between $H_2(\partial\Omega)$ and $H_2(\Omega) \oplus H_2(\mathbf{R}^3 - \Omega)$, again induced by the inclusions.

Our next tool is the relative homology exact sequence for the pair $(\mathbf{R}^3 - \Omega,\ \partial\Omega)$. The relevant portion reads

$$H_2(\partial\Omega) \xrightarrow{a} H_2(\mathbf{R}^3 - \Omega) \xrightarrow{b} H_2(\mathbf{R}^3 - \Omega, \partial\Omega) \xrightarrow{\delta} H_1(\partial\Omega) \xrightarrow{c} H_1(\mathbf{R}^3 - \Omega)$$

We first prove that the arrow marked δ is one-to-one. It suffices to show that b is the zero map. But the map a comes from the inclusion $\partial\Omega \hookrightarrow \mathbf{R}^3 - \Omega$, so a must be onto by the second Mayer-Vietoris argument above.

We have proved that δ is an isomorphism between $H_2(\mathbf{R}^3 - \Omega, \partial\Omega)$ and the image of δ inside $H_1(\partial\Omega)$. Yet the image of δ is the kernel of the arrow marked c above.

And by the first Mayer-Vietoris argument above, since c is induced by the inclusion $\partial\Omega \hookrightarrow \mathbf{R}^3 - \Omega$, the kernel of c is $H_1(\Omega) \subset H_1(\partial\Omega)$.

This completes the proof of the proposition.

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