Parametrized curves, examples and constructions

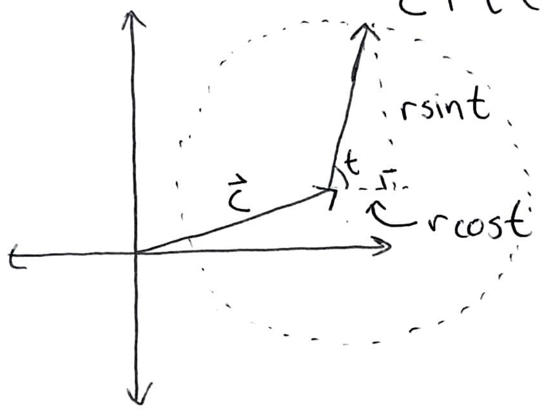
Recall that a parametrized curve is a map $\vec{\alpha}: \mathbb{R} \to \mathbb{R}^n$. We will now study some example curves.

Example. The circle of radius r with center $\vec{c} = (c_1, c_2)$ in $\mathbb{R}^2$ is described implicitly bey

$$(x - c_1)^2 + (y - c_2)^2 = r^2$$

We can parametrize this curve by

$$\vec{c} + r(\cos t, \sin t) = \vec{\alpha}(t)$$



Notice that

$$\vec{\alpha}(t) = (c_1 + r\cos t,\ c_2 + r\sin t)$$

obeys

$$(\alpha_1(t) - c_1)^2 + (\alpha_2(t) - c_2)^2 =$$

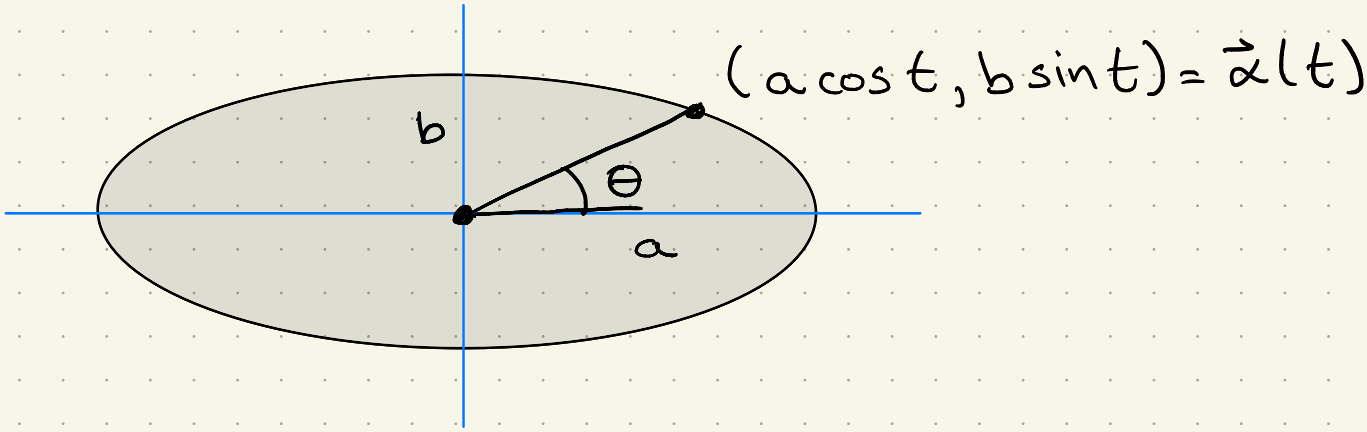
$$= (r\cos t)^2 + (r\sin t)^2$$

$$= r^2(\cos^2 t + \sin^2 t) = r^2,$$

but there is more information in the ~~para~~ parametrization $\vec{\alpha}(t)$ because it tells us when each point on the circle is reached.

Example 2. $\vec{\alpha}(t) = (c_1 + r\cos(t^2), c_2 + r\sin(t^2))$ also parametrizes the circle of radius r and center $\vec{c} = (c_1, c_2)$.

Example. The ellipse



We note that t is <u>not</u> the angle θ even though it's a natural guess.

Easy. Points on the ellipse satisfy

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Proof. Substituting in our parametrization

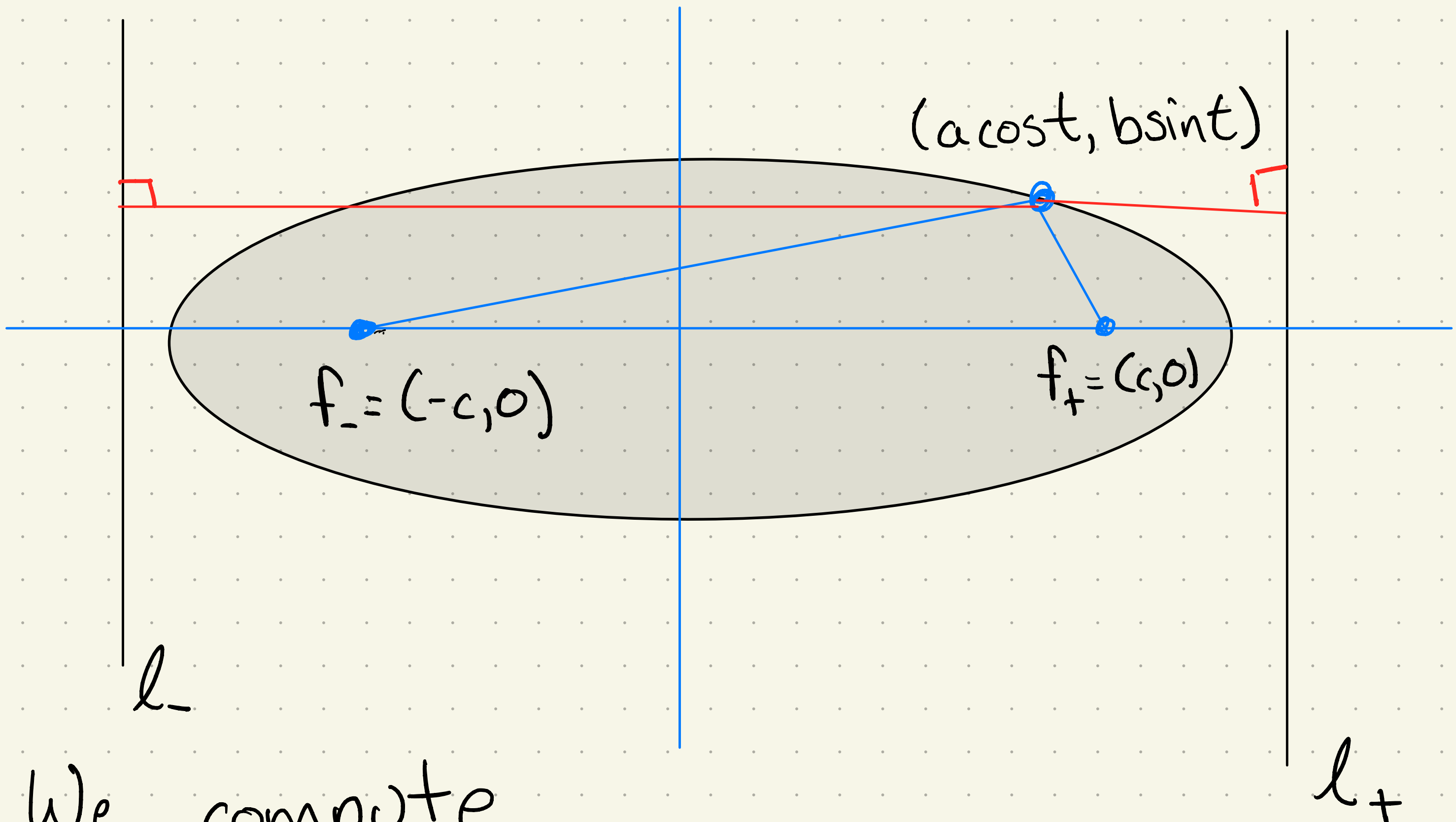
$$\frac{a^2\cos^2 t}{a^2} + \frac{b^2\sin^2 t}{b^2} = \cos^2 t + \sin^2 t$$

$$= 1$$

Harder. Let $c = \sqrt{a^2 - b^2}$ (assuming $a > b$) and $\vec{f}_- = (-c, 0)$, $f_+ = (c, 0)$. For any t,

$$\|\vec{\alpha}(t) - \vec{f}_-\| + \|\vec{\alpha}(t) - \vec{f}_+\| = 2a$$

Consider the lines $\left(\pm \frac{a^2}{c}, y\right)$.



We compute

$$\frac{\|\vec{\alpha}(t) - \vec{f}_+\|}{\|\vec{\alpha}(t) - \ell_+\|} = \frac{\sqrt{(a\cos t - c)^2 + b^2\sin^2 t}}{\left|\frac{a^2}{c} - a\cos t\right|}$$

Now

$$(a\cos t - c)^2 + b^2 \sin^2 t =$$

$$= a^2\cos^2 t - 2ac\cos t + c^2 + b^2\sin^2 t$$

$$= (a^2 - b^2)\cos^2 t - 2ac\cos t + c^2 + b^2$$

$$= c^2\cos^2 t - 2ac\cos t + a^2$$

$$= (c\cos t - a)^2$$

So we can write

$$\frac{\|\vec{\alpha}(t) - \vec{F}_+\|}{\|\vec{\alpha}(t) - \ell_+\|} = \frac{|a - c\cos t|}{\frac{a}{c}|a - c\cos t|} = \frac{c}{a}$$

By symmetry

$$\frac{\|\vec{\alpha}(t) - \vec{F}_-\|}{\|\vec{\alpha}(t) - \ell_-\|} = \frac{c}{a}$$ as well.

Therefore

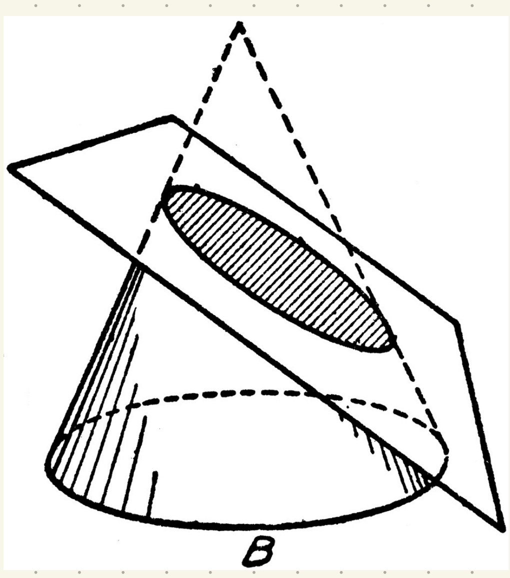
$$\|\vec{\alpha}(t) - \vec{f}_-\| + \|\vec{\alpha}(t) - \vec{f}_+\| =$$

$$= \frac{c}{a}\|\vec{\alpha}(t) - \ell_+\| + \frac{c}{a}\|\vec{\alpha}(t) - \ell_-\|$$

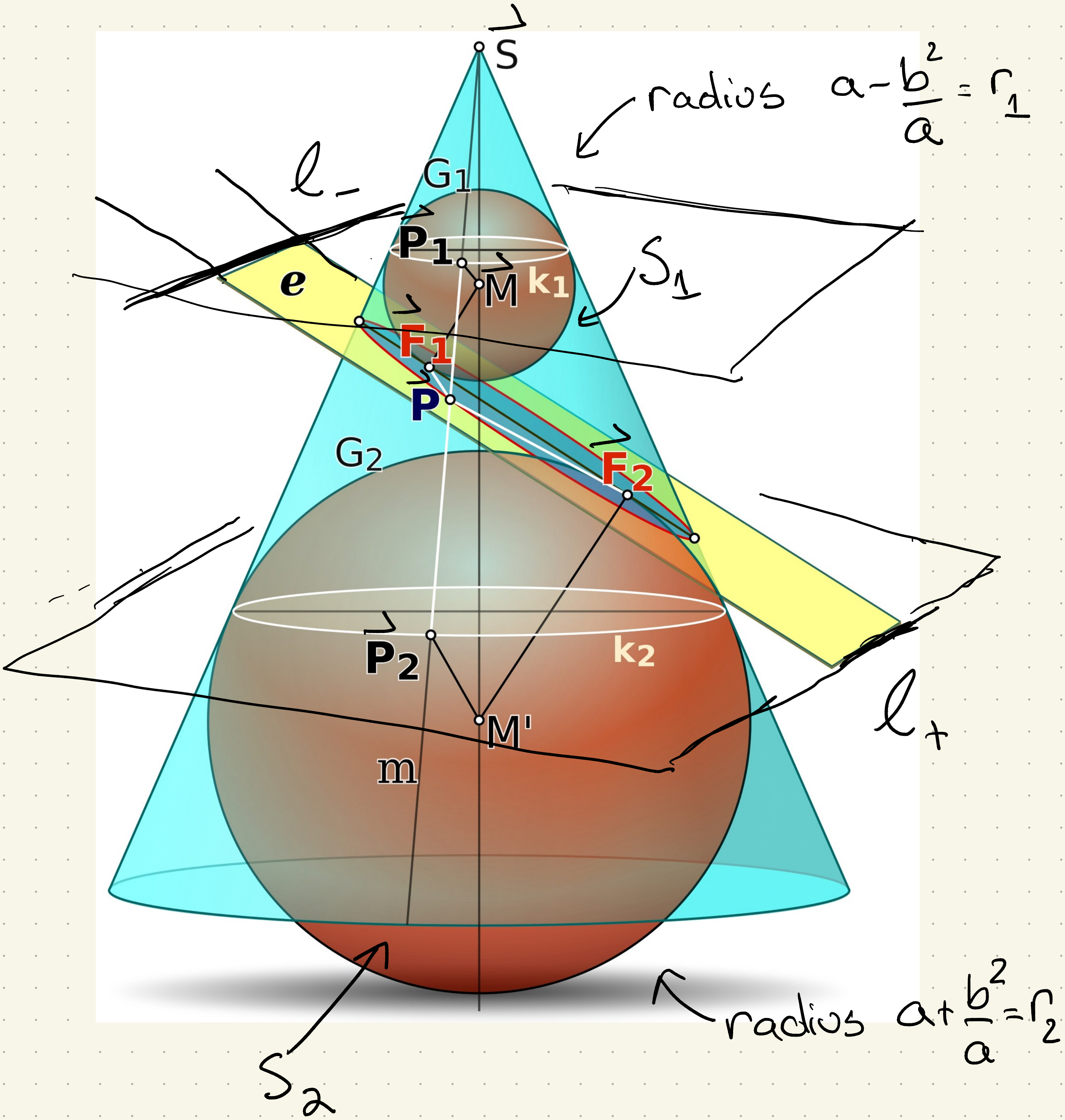
$$= \frac{c}{a}\|\ell_+ - \ell_-\|$$ ← since the vectors to the closest points on ℓ_+, ℓ_- are horizontal and hence colinear

$$= \frac{c}{a} \cdot \frac{2a^2}{c} = 2a.$$ ▨

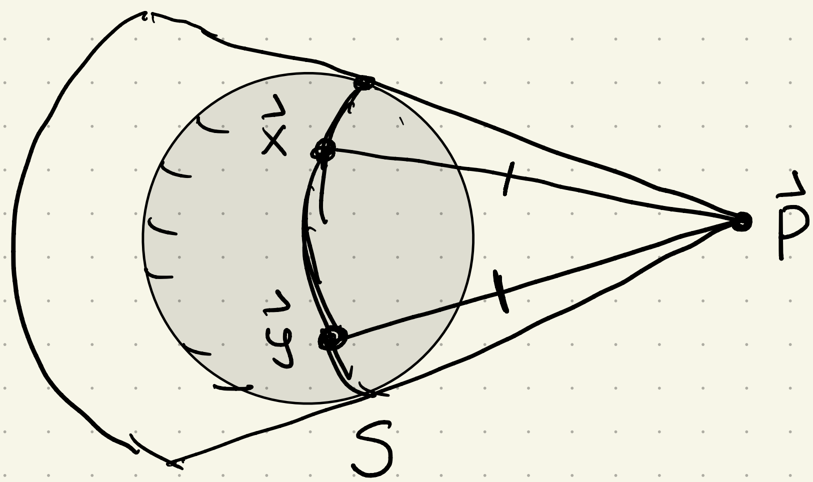
We see ellipses around us all the time as intersections of cones and planes.



To prove that the intersection is an ellipse, consider the Dandelin spheres



Useful fact.



Suppose $\vec{p}$ is outside a sphere S, $\vec{x}$ and $\vec{y}$ are on S, and $\overrightarrow{px}, \overrightarrow{py}$ are tangent to S at $\vec{x}, \vec{y}$. Then $\|\vec{p}-\vec{x}\| = \|\vec{p}-\vec{y}\|$.

Proof that intersection is ellipse. Suppose $\vec{P}$ is on the curve. Draw the line m on the cone through the vertex $\vec{S}$ and $\vec{P}$, suppose m intersects the circles at $\vec{p}_1, \vec{p}_2$

Now $\vec{p}\vec{p}_1$ and $\vec{p}\vec{f}_1$ are tangent to S_1, so $\|\vec{p}-\vec{p}_1\| = \|\vec{p}-\vec{f}_1\|$ by <u>fact</u>. A similar argument shows $\|\vec{p}-\vec{p}_2\| = \|\vec{p}-\vec{f}_2\|$. But then

$$\|\vec{p}-\vec{f}_1\| + \|\vec{p}-\vec{f}_2\| = \|\vec{p}-\vec{p}_1\| + \|\vec{p}-\vec{p}_2\|$$

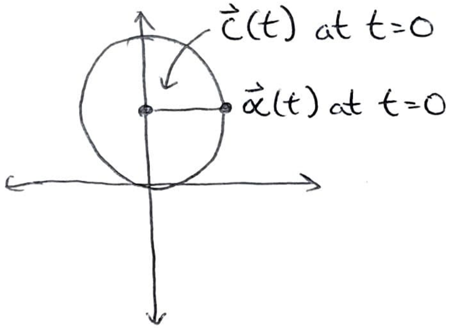
$$= \|\vec{p}_1-\vec{p}_2\|$$

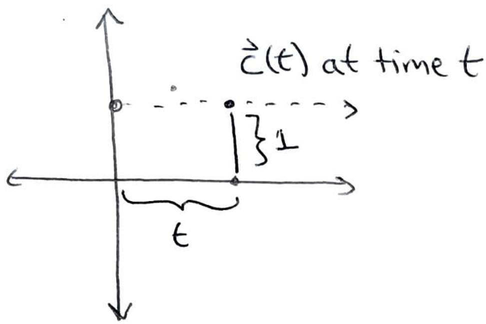
$P_1 P P_2$ is a straight line

and $\|\vec{p}_1-\vec{p}_2\|$ is the distance between the parallel circles K_1, K_2 which does not depend on $\vec{p}$. $\boxtimes$

We can make some beautiful curves by combining sines and cosines.

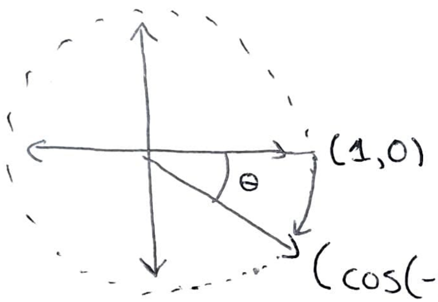
Example. A unit circle starts with center at $(0,1)$ and rolls along the pos. x axis. Parametrize the path of a point starting at $(1,1)$.





If the center of the circle is given by $\vec{c}(t)$, we can assume that the circle is rolling to the right at unit speed, so $\vec{c}(t) = (t, 1)$.

However, if ~~th~~ a unit circle has rolled t units forward, it has turned by an angle of t radians... in the <u>clockwise</u> direction.

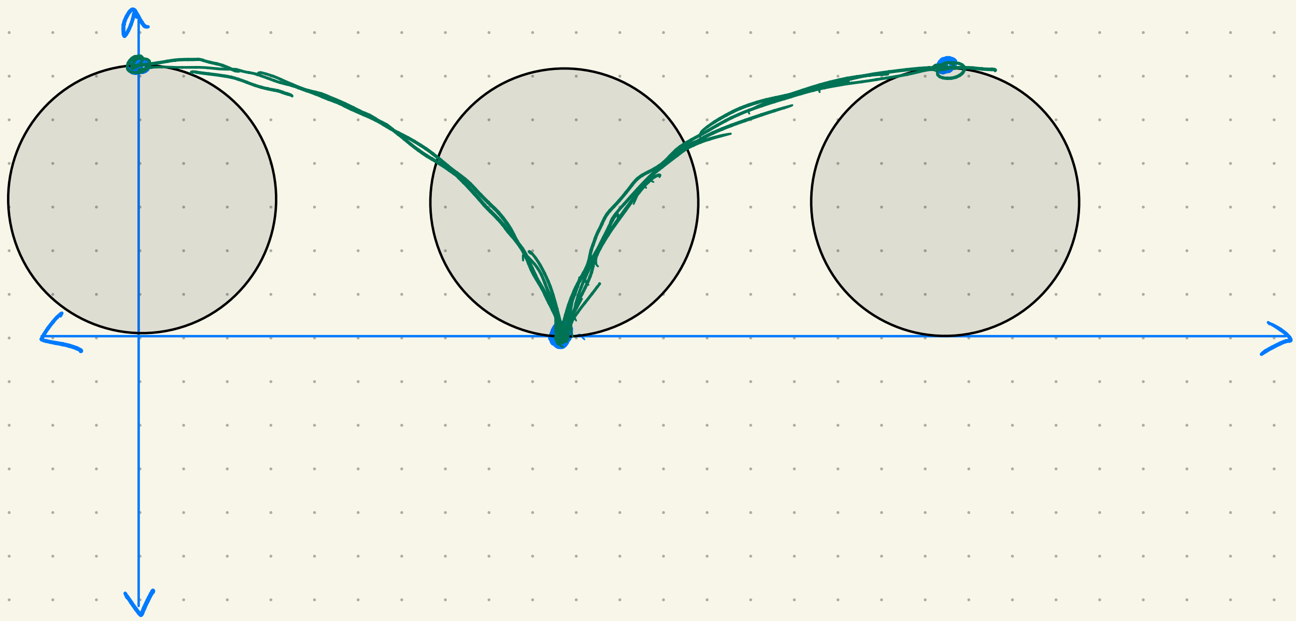


$(\cos(-\theta), \sin(-\theta)) = (\cos\theta, -\sin\theta)$

This rotation carries the point at $(1,0)$ ~~to~~ ~~the~~ (relative to the center) to the point at $(\cos\theta, -\sin\theta)$ (relative to the center).

Adding these together:

$$\vec{\alpha}(t) = (t + \cos t, 1 - \sin t)$$



Note: It's not easy to graph this curve from the parametrization.

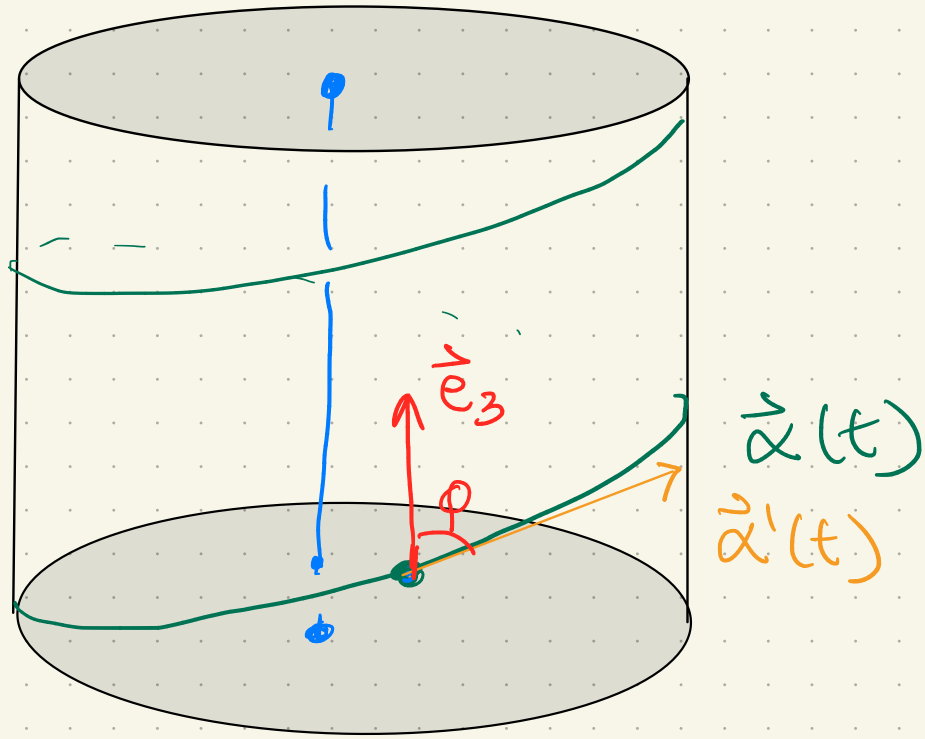
Example. The helix. Given $\theta \in [0, \pi/2]$, $r > 0$

$$\vec{\alpha}(t) = (r\cos t, r\sin t, t\, r\tan\theta)$$

Suppose that $\beta(t) = (0, 0, t\, r\tan\theta)$ (This is the z-axis.) We see

$$\|\vec{\alpha}(t) - \beta(t)\|^2 = r^2\cos^2 t + r^2\sin^2 t = r^2.$$

So $\vec{\alpha}(t)$ is on cylinder of radius r around the z-axis.



$$\vec{\alpha}'(t) = (-r\sin t, r\cos t, r\tan\theta)$$

We start by computing

$$\cos\varphi = \frac{\langle \vec{\alpha}'(t), \vec{e}_3 \rangle}{\|\vec{\alpha}'(t)\| \, \|\vec{e}_3\|} = \frac{r\tan\theta}{\sqrt{r^2 + r^2\tan^2\theta}}$$

And recalling

$$\sin^2\theta + \cos^2\theta = 1 \Rightarrow \tan^2\theta + 1 = \sec^2\theta$$

so (since $r > 0$, $\theta \in [0, \pi/2]$)

$$\sqrt{r^2 + r^2\tan^2\theta} = r\sec\theta$$

and so

$$\frac{r\tan\theta}{\sqrt{r^2 + r^2\tan^2\theta}} = \frac{r\tan\theta}{r\sec\theta} = \sin\theta$$

But this means that

$$\cos\varphi = \sin\theta$$

Using $\cos\varphi = \sin(\pi/2 - \varphi)$, we see $\theta = \pi/2 - \varphi =$ angle with horizontal.

This is called the <u>pitch angle</u> of the helix.

Note: Nuts and bolts have threads cut in a helical pattern. These are usually specified by diameter and "tpi" or threads per inch.

$$tpi = \frac{\#\text{ complete revolutions in xy plane}}{1 \text{ inch of z-axis}}$$

$$= \frac{t/2\pi}{t\,r\tan\theta/1} = \frac{\cot\theta}{2\pi r}$$

depends on r <u>and</u> θ

Compare #8-32 and #6-32 screws.

Milling, threads and industrial revolution.