Parametrized curves, examples and constructions

Recall that a parametrized curve is a map $\hat{\alpha}: \mathbb{R} \to \mathbb{R}^2$. We will now study some example curves.

Example. The circle of radius $r$ with center $\hat{c} = (c_1, c_2)$ in $\mathbb{R}^2$ is described implicitly by

$$(x - c_1)^2 + (y - c_2)^2 = r^2$$

We can parametrize this curve by

$$\hat{\alpha} + \Gamma (\cos t, \sin t) = \hat{\alpha}(t)$$
Notice that
\[ \hat{\alpha}(t) = (c_1 + r\cos t, c_2 + r\sin t) \]
obeys
\[ (\alpha_1(t) - c_1)^2 + (\alpha_2(t) - c_2)^2 = \]
\[ = (r\cos t)^2 + (r\sin t)^2 \]
\[ = r^2 (\cos^2 t + \sin^2 t) = r^2, \]
but there is more information in
the parametrization \( \hat{\alpha}(t) \) because it tells us when each point on the circle is reached.

Example 2. \( \hat{\alpha}(t) = (c_1 + r\cos(t^2), c_2 + r\sin(t^2)) \)
also parametrizes the circle of radius \( r \) and center \( \hat{c} = (c_1, c_2) \).
Example. The ellipse

\[(a \cos t, b \sin t) = \mathbf{x}(t)\]

We note that \( t \) is not the angle \( \theta \) even though it's a natural guess.

Easy. Points on the ellipse satisfy

\[\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\]

Proof. Substituting in our parametrization

\[\frac{a^2 \cos^2 t}{a^2} + \frac{b^2 \sin^2 t}{b^2} = \cos^2 t + \sin^2 t = 1\]
Harder. Let $c = \sqrt{a^2 - b^2}$ (assuming $a > b$) and $\vec{f}_- = (-c,0)$, $\vec{f}_+ = (c,0)$. For any $t$,

$$||\vec{a}(t) - \vec{f}_-|| + ||\vec{a}(t) - \vec{f}_+|| = 2a$$

Consider the lines $(\pm \frac{a^2}{c}, y)$.

We compute

$$||\vec{a}(t) - \vec{p}_+|| = \sqrt{(ac \cos t - c)^2 + b^2 \sin^2 t}$$

$$||\vec{a}(t) - \ell_t|| = \left| \frac{a^2}{c} - a \cos t \right|$$
Now

\[(a \cos t - c)^2 + b^2 \sin^2 t =\]

\[= a^2 \cos^2 t - 2ac \cos t + c^2 + b^2 \sin^2 t\]

\[= (a^2 - b^2) \cos^2 t - 2ac \cos t + c^2 + b^2\]

\[= c^2 \cos^2 t - 2ac \cos t + a^2\]

\[= (c \cos t - a)^2\]

So we can write

\[
\frac{\| \vec{\alpha}(t) - \vec{P}_+ \|}{\| \vec{\alpha}(t) - \vec{l}_+ \|} = \frac{|a - c \cos t|}{\frac{a}{c} |a - c \cos t|} = \frac{c}{a}
\]

By symmetry

\[
\frac{\| \vec{\alpha}(t) - \vec{P}_- \|}{\| \vec{\alpha}(t) - \vec{l}_- \|} = \frac{c}{a} \quad \text{as well.}
\]
Therefore

\[ \| \vec{a}(t) - \vec{f}_- \| + \| \vec{a}(t) - \vec{f}_+ \| = \]

\[ = \frac{c}{a} \| \vec{a}(t) - l_+ \| + \frac{c}{a} \| \vec{a}(t) - l_- \| \]

\[ = \frac{c}{a} \| l_+ - l_- \| \leftarrow \text{since the vectors to the closest points on } l_+, l_- \text{ are horizontal and hence colinear} \]

\[ = \frac{c}{a} \cdot \frac{2a^2}{c} = 2a. \]

We see ellipses around us all the time as intersections of cones and planes.
To prove that the intersection is an ellipse, consider the Dandelin spheres.
Useful fact.

Suppose \( \vec{p} \) is outside a sphere \( S \), \( \vec{x} \) and \( \vec{y} \) are on \( S \), and \( \vec{p} \vec{x}, \vec{p} \vec{y} \) are tangent to \( S \) at \( \vec{x}, \vec{y} \). Then \( ||\vec{p} - \vec{x}|| = ||\vec{p} - \vec{y}|| \).

Proof that intersection is ellipse.
Suppose \( \vec{P} \) is on the curve. Draw the line \( m \) on the cone through the vertex \( \vec{S} \) and \( \vec{P} \), suppose \( m \) intersects the circles at \( \vec{P}_1, \vec{P}_2 \).
Now \( \hat{p} \hat{p}_1 \) and \( \hat{p} \hat{p}_1 \) are tangent to \( S_1 \), so \( \| \hat{p} - \hat{p}_1 \| = \| \hat{p} - \hat{p}_2 \| \) by fact. A similar argument shows \( \| \hat{p} - \hat{p}_2 \| = \| \hat{p} - \hat{p}_2 \| \). But then

\[
\| \hat{p} - \hat{p}_1 \| + \| \hat{p} - \hat{p}_2 \| = \| \hat{p} - \hat{p}_1 \| + \| \hat{p} - \hat{p}_2 \| = \| \hat{p}_1 - \hat{p}_2 \|
\]

\( P_1 P P_2 \) is a straight line and \( \| \hat{p}_1 - \hat{p}_2 \| \) is the distance between the parallel circles \( K_1, K_2 \) which does not depend on \( \hat{p} \). □
We can make some beautiful curves by combining sines and cosines.

Example. A unit circle starts with center at \((0,1)\) and rolls along the positive \(x\) axis. Parametrize the path of a point starting at \((1,1)\).

If the center of the circle is given by \(\vec{c}(t)\), we can assume that the circle is rolling to the right at unit speed, so \(\vec{c}(t) = (t, 1)\).
However, if a unit circle has rolled $t$ units forward, it has turned by an angle of $t$ radians... in the **clockwise** direction.

\[
(\cos(-\theta), \sin(-\theta)) = (\cos \theta, -\sin \theta)
\]

This rotation carries the point at (1,0) to \( (\cos \theta, -\sin \theta) \) (relative to the center) to the point at \( (\cos \theta, -\sin \theta) \) (relative to the center).

Adding these together:

\[
\vec{x}(t) = (t + \cos \frac{\theta}{2}, 1 - \sin t)
\]
Note: It's not easy to graph this curve from the parametrization.

Example. The helix. Given $\Theta \in [0,\pi/2], r > 0$

$$\vec{\alpha}(t) = (r \cos t, r \sin t, t \cdot r \tan \Theta)$$

Suppose that $\beta(t) = (0,0,t \cdot r \tan \Theta)$ (This is the $z$-axis.) We see

$$\|\vec{\alpha}(t) - \beta(t)\|^2 = r^2 \cos^2 t + r^2 \sin^2 t = r^2.$$
So $\mathbf{x}(t)$ is on cylinder of radius $r$ around the $z$-axis.

\[ \mathbf{x}'(t) = (-r \sin t, r \cos t, r \tan \theta) \]

We start by computing

\[ \cos \phi = \frac{\langle \mathbf{x}'(t), \hat{e}_3 \rangle}{\| \mathbf{x}'(t) \| \| \hat{e}_3 \|} = \frac{r \tan \theta}{\sqrt{r^2 + r^2 \tan^2 \theta}} \]
And recalling
\[ \sin^2\theta + \cos^2\theta = 1 \Rightarrow \tan^2\theta + 1 = \sec^2\theta \]
so (since \( r > 0 \), \( \theta \in [0, \pi/2] \))
\[ \sqrt{r^2 + r^2\tan^2\theta} = r \sec\theta \]
and so
\[ \frac{r \tan\theta}{\sqrt{r^2 + r^2\tan^2\theta}} = \frac{r \tan\theta}{r \sec\theta} = \sin\theta \]

But this means that
\[ \cos \phi = \sin \theta \]
Using \( \cos \phi = \sin(\pi/2 - \phi) \), we see \( \Theta = \pi/2 - \phi = \text{angle with horizontal} \).
This is called the pitch angle of the helix.
Note: Nuts and bolts have threads cut in a helical pattern. These are usually specified by diameter and "tpi" or threads per inch.

$$t_{pi} = \frac{\text{# complete revolutions in xy plane}}{1 \text{ inch of z-axis}}$$

$$= \frac{t/2\pi}{t \tan \theta / \sqrt{1}} = \frac{\cot \theta}{2\pi r}$$

↑ depends on r and θ

Compare #8-32 and #6-32 screws.
Milling, threads and industrial revolution.