Chapter 6

Fundamental Graphs

We will bound and derive the eigenvalues of the Laplacian matrices of some fundamental graphs, including complete graphs, star graphs, ring graphs, path graphs, and products of these that yield grids and hypercubes. As all these graphs are connected, they all have eigenvalue zero with multiplicity one. We will have to do some work to compute the other eigenvalues.

We will see in Part IV that the Laplacian eigenvalues that reveal the most about a graph are the smallest and largest ones. To interpret the smallest eigenvalues, we will exploit a relation between λ_2 and the isoperimetric ratio of a graph that is derived in Chapter 20, and which we state here for convenience:

For every $S \subset V$,

$$\theta(S) \geq \lambda_2(1 - s),$$

where $s = |S| \,/\, |V|$ and

$$\theta(S) \stackrel{\text{def}}{=} \frac{|\partial(S)|}{|S|}$$

is the *isoperimetric ratio* of S.

6.1 The complete graph

The complete graph on n vertices, K_n, has edge set $\{(a,b) : a \neq b\}$.

Lemma 6.1.1. *The Laplacian of K_n has eigenvalue 0 with multiplicity 1 and n with multiplicity $n-1$.*

Proof. To compute the non-zero eigenvalues, let $\boldsymbol{\psi}$ be any non-zero vector orthogonal to the all-1s vector, so

$$\sum_a \boldsymbol{\psi}(a) = 0. \tag{6.1}$$

We now compute the first coordinate of $\boldsymbol{L}_{K_n}\boldsymbol{\psi}$. Using (3.3), the expression for the action of the Laplacian as an operator, we find

$$(\boldsymbol{L}_{K_n}\boldsymbol{\psi})(1) = \sum_{v\geq 2}(\boldsymbol{\psi}(1) - \boldsymbol{\psi}(b)) = (n-1)\boldsymbol{\psi}(1) - \sum_{v=2}^{n}\boldsymbol{\psi}(b) = n\boldsymbol{\psi}(1), \quad \text{by (6.1)}.$$

As the choice of coordinate was arbitrary, we have $\boldsymbol{L}\boldsymbol{\psi} = n\boldsymbol{\psi}$. So, every vector orthogonal to the all-1s vector is an eigenvector of eigenvalue n. □

Alternative approach. Observe that $\boldsymbol{L}_{K_n} = n\boldsymbol{I} - \mathbf{1}\mathbf{1}^T$. □

We often think of the Laplacian of the complete graph as being a scaling of the identity. For every $\boldsymbol{x}$ orthogonal to the all-1s vector, $\boldsymbol{L}\boldsymbol{x} = n\boldsymbol{x}$.

Now, let's see how our bound on the isoperimetric ratio works out. Let $S \subset [n]$. Every vertex in S has $n - |S|$ edges connecting it to vertices not in S. So,

$$\theta(S) = \frac{|S|(n - |S|}{|S|} = n - |S| = \lambda_2(\boldsymbol{L}_{K_n})(1-s),$$

where $s = |S|/n$. Thus, Theorem 20.1.1 is sharp for the complete graph.

6.2 The star graphs

The star graph on n vertices S_n has edge set $\{(1,a) : 2 \leq a \leq n\}$.

To determine the eigenvalues of S_n, we first observe that each vertex $a \geq 2$ has degree 1, and that each of these degree-one vertices has the same neighbor. Whenever two degree-one vertices share the same neighbor, they provide an eigenvector of eigenvalue 1.

Lemma 6.2.1. *Let $G = (V,E)$ be a graph, and let a and b be vertices of degree one that are both connected to another vertex c. Then, the vector $\boldsymbol{\psi} = \boldsymbol{\delta}_a - \boldsymbol{\delta}_b$ is an eigenvector of $\boldsymbol{L}_G$ of eigenvalue 1.*

Proof. Just multiply $\boldsymbol{L}_G$ by $\boldsymbol{\psi}$, and check (using (3.3)) vertex-by-vertex that it equals $\boldsymbol{\psi}$. □

As eigenvectors of different eigenvalues are orthogonal, this implies that $\boldsymbol{\psi}(a) = \boldsymbol{\psi}(b)$ for every eigenvector with eigenvalue different from 1.

Lemma 6.2.2. *The graph S_n has eigenvalue 0 with multiplicity 1, eigenvalue 1 with multiplicity $n-2$, and eigenvalue n with multiplicity 1.*

Proof. Applying Lemma 6.2.1 to vertices i and $i+1$ for $2 \leq i < n$, we find $n-2$ linearly independent eigenvectors of the form $\boldsymbol{\delta}_i - \boldsymbol{\delta}_{i+1}$, all with eigenvalue 1. As 0 is also an eigenvalue, only one eigenvalue remains to be determined.

Recall that the trace of a matrix equals both the sum of its diagonal entries and the sum of its eigenvalues. We know that the trace of $\boldsymbol{L}_{S_n}$ is $2n-2$, and we have identified $n-1$ eigenvalues that sum to $n-2$. So, the remaining eigenvalue must be n.

To determine the corresponding eigenvector, recall that it must be orthogonal to the other eigenvectors we have identified. This tells us that it must have the same value at each of the points of the star. Let this value be 1, and let x be the value at vertex 1. As the eigenvector is orthogonal to the constant vectors, it must be that

$$(n-1)+x=0,$$

so $x=-(n-1)$. □

6.3 Products of graphs

We now define a product on graphs. If we apply this product to two paths, we obtain a grid. If we apply it repeatedly to one edge, we obtain a hypercube.

Definition 6.3.1. *Let $G=(V,E,v)$ and $H=(W,F,w)$ be weighted graphs. Then $G\times H$ is the graph with vertex set $V\times W$ and edge set*

$$\Big((a,b),(\widehat{a},b)\Big) \text{ with weight } v_{a,\widehat{a}}, \text{ where } (a,\widehat{a})\in E \text{ and}$$
$$\Big((a,b),(a,\widehat{b})\Big) \text{ with weight } w_{b,\widehat{b}}, \text{ where } (b,\widehat{b})\in F.$$

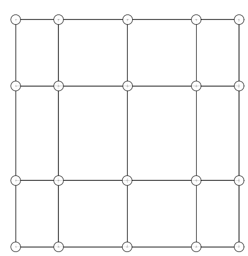

Figure 6.1: An m-by-n grid graph is the product of a path on m vertices with a path on n vertices. This is a drawing of a 5-by-4 grid made using Hall's algorithm.

Theorem 6.3.2. *Let $G=(V,E,v)$ and $H=(W,F,w)$ be weighted graphs with Laplacian eigenvalues $\lambda_1,\dots,\lambda_n$ and $\mu_1,\dots,\mu_m$, and eigenvectors $\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_n$ and $\boldsymbol{\beta}_1,\dots,\boldsymbol{\beta}_m$, respectively. Then, for each $1\le i\le n$ and $1\le j\le m$, $G\times H$ has an eigenvector $\boldsymbol{\gamma}_{i,j}$ of eigenvalue $\lambda_i+\mu_j$ such that*

$$\boldsymbol{\gamma}_{i,j}(a,b)=\boldsymbol{\alpha}_i(a)\boldsymbol{\beta}_j(b).$$

Proof. Let $\boldsymbol{\alpha}$ be an eigenvector of L_G of eigenvalue λ, let $\boldsymbol{\beta}$ be an eigenvector of L_H of eigenvalue μ, and let $\boldsymbol{\gamma}$ be defined as above.

To see that $\boldsymbol{\gamma}$ is an eigenvector of eigenvalue $\lambda + \mu$, we compute

$$\begin{aligned}
(\boldsymbol{L}\boldsymbol{\gamma})(a,b) &= \sum_{(a,\widehat{a}) \in E} v_{a,\widehat{a}} \left(\boldsymbol{\gamma}(a,b) - \boldsymbol{\gamma}(\widehat{a},b)\right) + w_{b,\widehat{b}} \sum_{(b,\widehat{b}) \in F} \left(\boldsymbol{\gamma}(a,b) - \boldsymbol{\gamma}(a,\widehat{b})\right) \\
&= \sum_{(a,\widehat{a}) \in E} v_{a,\widehat{a}} \left(\boldsymbol{\alpha}(a)\boldsymbol{\beta}(b) - \boldsymbol{\alpha}(\widehat{a})\boldsymbol{\beta}(b)\right) + \sum_{(b,\widehat{b}) \in F} w_{b,\widehat{b}} \left(\boldsymbol{\alpha}(a)\boldsymbol{\beta}(b) - \boldsymbol{\alpha}(a)\boldsymbol{\beta}(\widehat{b})\right) \\
&= \sum_{(a,\widehat{a}) \in E} v_{a,\widehat{a}} \boldsymbol{\beta}(b) \left(\boldsymbol{\alpha}(a) - \boldsymbol{\alpha}(\widehat{a})\right) + \sum_{(b,\widehat{b}) \in F} w_{b,\widehat{b}} \boldsymbol{\alpha}(a) \left(\boldsymbol{\beta}(b) - \boldsymbol{\beta}(\widehat{b})\right) \\
&= \sum_{(a,\widehat{a}) \in E} \boldsymbol{\beta}(b) \lambda \boldsymbol{\alpha}(a) + \sum_{(b,\widehat{b}) \in F} \boldsymbol{\alpha}(a) \mu \boldsymbol{\beta}(b) \\
&= (\lambda + \mu)(\boldsymbol{\alpha}(a)\boldsymbol{\beta}(b)).
\end{aligned}$$

□

An alternative approach to defining the graph product and proving Theorem 6.3.2 is via Kronecker products. $G \times H$ is the graph with Laplacian matrix

$$(\boldsymbol{L}_G \otimes \boldsymbol{I}_W) + (\boldsymbol{I}_V \otimes \boldsymbol{L}_H).$$

6.3.1 The Hypercube

The d-dimensional hypercube graph, H_d, is the graph with vertex set $\{0,1\}^d$, with edges between vertices whose names differ in exactly one bit. The hypercube may also be expressed as the product of the one-edge graph with itself $d - 1$ times.

Let H_1 be the graph with vertex set $\{0,1\}$ and one edge between those vertices. It's Laplacian matrix has eigenvalues 0 and 2. As $H_d = H_{d-1} \times H_1$, we may use this to calculate the eigenvalues and eigenvectors of H_d for every d.

The eigenvectors of H_1 are

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

with eigenvalues 0 and 2, respectively. Thus, if $\boldsymbol{\psi}$ is an eigenvector of H_{d-1} with eigenvalue λ, then

$$\begin{pmatrix} \boldsymbol{\psi} \\ \boldsymbol{\psi} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \boldsymbol{\psi} \\ -\boldsymbol{\psi} \end{pmatrix},$$

are eigenvectors of H_d with eigenvalues λ and $\lambda + 2$, respectively. This means that H_d has eigenvalue $2i$ for each $0 \leq i \leq d$ with multiplicity $\binom{d}{i}$. Moreover, each eigenvector of H_d can be identified with a vector $\boldsymbol{y} \in \{0,1\}^d$:

$$\boldsymbol{\psi}_{\boldsymbol{y}}(\boldsymbol{x}) = (-1)^{\boldsymbol{y}^T \boldsymbol{x}},$$

where $\boldsymbol{x} \in \{0,1\}^d$ ranges over the vertices of H_d. Each $\boldsymbol{y} \in \{0,1\}^{d-1}$ indexing an eigenvector of H_{d-1} leads to the eigenvectors of H_d indexed by $(\boldsymbol{y}, 0)$ and $(\boldsymbol{y}, 1)$.

Using Theorem 20.1.1 and the fact that $\lambda_2(H_d) = 2$, we can immediately prove the following isoperimetric theorem for the hypercube.

Corollary 6.3.3.

$$\theta_{H_d} \geq 1.$$

In particular, for every set of at most half the vertices of the hypercube, the number of edges on the boundary of that set is at least the number of vertices in that set.

This result is tight, as you can see by considering one face of the hypercube, such as all the vertices whose labels begin with 0. It is possible to prove this by more concrete combinatorial means. In fact, very precise analyses of the isoperimetry of sets of vertices in the hypercube can be obtained. See [Har76] or [Bol86].

6.4 Bounds on λ_2 by test vectors

If we can guess an approximation of $\boldsymbol{\psi}_2$, we can often plug it in to the Laplacian quadratic form to obtain a good upper bound on λ_2. The Courant-Fischer Theorem tells us that every vector $\boldsymbol{v}$ orthogonal to $\mathbf{1}$ provides an upper bound on λ_2:

$$\lambda_2 \leq \frac{\boldsymbol{v}^T \boldsymbol{L} \boldsymbol{v}}{\boldsymbol{v}^T \boldsymbol{v}}.$$

When we use a vector $\boldsymbol{v}$ in this way, we call it a *test vector.*

Let's see what a test vector can tell us about λ_2 of a path graph on n vertices. I would like to use the vector that assigns i to vertex a as a test vector, but it is not orthogonal to $\mathbf{1}$. So, we will use the next best thing. Let $\boldsymbol{x}$ be the vector such that $\boldsymbol{x}(a) = (n+1) - 2a$, for $1 \leq a \leq n$. This vector satisfies $\boldsymbol{x} \perp \mathbf{1}$, so

$$\begin{aligned}
\lambda_2(P_n) &\leq \frac{\sum_{1 \leq a < n} (x(a) - x(a+1))^2}{\sum_a x(a)^2} \\
&= \frac{\sum_{1 \leq a < n} 2^2}{\sum_a (n+1-2a)^2} \\
&= \frac{4(n-1)}{(n+1)n(n-1)/3} && \text{(clearly, the denominator is } n^3/c \text{ for some } c) \\
&= \frac{12}{n(n+1)}. && (6.2)
\end{aligned}$$

We will soon see that this bound is of the right order of magnitude. Thus, Theorem 20.1.1 does not provide a good bound on the isoperimetric ratio of the path graph. The isoperimetric ratio is minimized by the set $S = \{1, \ldots, n/2\}$, which has $\theta(S) = 2/n$. However, the upper bound