where $\boldsymbol{x} \in \{0,1\}^d$ ranges over the vertices of H_d. Each $\boldsymbol{y} \in \{0,1\}^{d-1}$ indexing an eigenvector of H_{d-1} leads to the eigenvectors of H_d indexed by $(\boldsymbol{y}, 0)$ and $(\boldsymbol{y}, 1)$.

Using Theorem 20.1.1 and the fact that $\lambda_2(H_d) = 2$, we can immediately prove the following isoperimetric theorem for the hypercube.

Corollary 6.3.3.

$$\theta_{H_d} \geq 1.$$

In particular, for every set of at most half the vertices of the hypercube, the number of edges on the boundary of that set is at least the number of vertices in that set.

This result is tight, as you can see by considering one face of the hypercube, such as all the vertices whose labels begin with 0. It is possible to prove this by more concrete combinatorial means. In fact, very precise analyses of the isoperimetry of sets of vertices in the hypercube can be obtained. See [Har76] or [Bol86].

6.4 Bounds on λ_2 by test vectors

If we can guess an approximation of $\boldsymbol{\psi}_2$, we can often plug it in to the Laplacian quadratic form to obtain a good upper bound on λ_2. The Courant-Fischer Theorem tells us that every vector $\boldsymbol{v}$ orthogonal to $\mathbf{1}$ provides an upper bound on λ_2:

$$\lambda_2 \leq \frac{\boldsymbol{v}^T \boldsymbol{L} \boldsymbol{v}}{\boldsymbol{v}^T \boldsymbol{v}}.$$

When we use a vector $\boldsymbol{v}$ in this way, we call it a *test vector.*

Let's see what a test vector can tell us about λ_2 of a path graph on n vertices. I would like to use the vector that assigns i to vertex a as a test vector, but it is not orthogonal to $\mathbf{1}$. So, we will use the next best thing. Let $\boldsymbol{x}$ be the vector such that $\boldsymbol{x}(a) = (n+1) - 2a$, for $1 \leq a \leq n$. This vector satisfies $\boldsymbol{x} \perp \mathbf{1}$, so

$$\begin{aligned}
\lambda_2(P_n) &\leq \frac{\sum_{1 \leq a < n} (x(a) - x(a+1))^2}{\sum_a x(a)^2} \\
&= \frac{\sum_{1 \leq a < n} 2^2}{\sum_a (n+1-2a)^2} \\
&= \frac{4(n-1)}{(n+1)n(n-1)/3} && \text{(clearly, the denominator is } n^3/c \text{ for some } c) \\
&= \frac{12}{n(n+1)}. && (6.2)
\end{aligned}$$

We will soon see that this bound is of the right order of magnitude. Thus, Theorem 20.1.1 does not provide a good bound on the isoperimetric ratio of the path graph. The isoperimetric ratio is minimized by the set $S = \{1, \ldots, n/2\}$, which has $\theta(S) = 2/n$. However, the upper bound provided by Theorem 20.1.1 is of the form c/n. Cheeger's inequality, which appears in Chapter 21, will tell us that the error of this approximation can not be worse than quadratic.

The Courant-Fischer theorem is not as helpful when we want to prove lower bounds on λ_2. To prove lower bounds, we need the form with a maximum on the outside, which gives

$$\lambda_2 \geq \max_{S:\dim(S)=n-1} \min_{\boldsymbol{v} \in S} \frac{\boldsymbol{v}^T \boldsymbol{L} \boldsymbol{v}}{\boldsymbol{v}^T \boldsymbol{v}}.$$

This is not too helpful, as it is difficult to prove lower bounds on

$$\min_{\boldsymbol{v} \in S} \frac{\boldsymbol{v}^T \boldsymbol{L} \boldsymbol{v}}{\boldsymbol{v}^T \boldsymbol{v}}$$

over a space S of large dimension. We will see a technique that lets us prove such lower bounds next lecture.

But, first we compute the eigenvalues and eigenvectors of the path graph exactly.

6.5 The Ring Graph

The ring graph on n vertices, R_n, may be viewed as having a vertex set corresponding to the integers modulo n. In this case, we view the vertices as the numbers 0 through $n-1$, with edges $(a, a+1)$, computed modulo n.

Lemma 6.5.1. *The Laplacian of R_n has eigenvectors*

$$\begin{aligned} \boldsymbol{x}_k(a) &= \cos(2\pi k a/n), \text{ and} \\ \boldsymbol{y}_k(a) &= \sin(2\pi k a/n), \end{aligned}$$

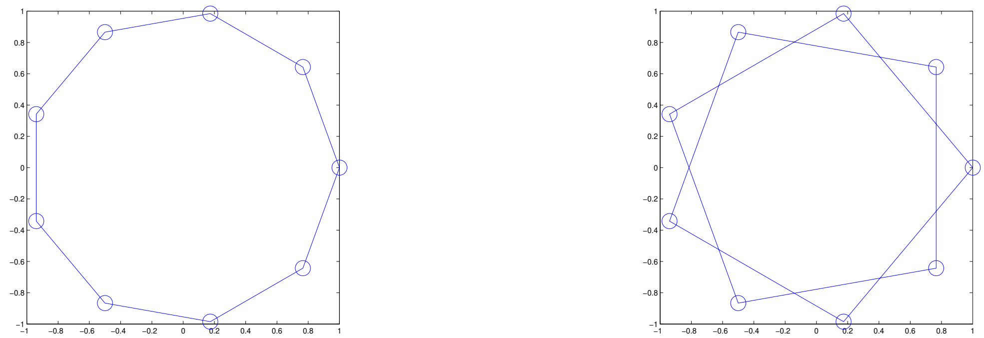
for $0 \leq k \leq n/2$, ignoring $\boldsymbol{y}_0$ which is the all-zero vector, and for even n ignoring $\boldsymbol{y}_{n/2}$ for the same reason. Eigenvectors $\boldsymbol{x}_k$ and $\boldsymbol{y}_k$ have eigenvalue $2 - 2\cos(2\pi k/n)$.

Note that $\boldsymbol{x}_0$ is the all-ones vector. When n is even, we only have $\boldsymbol{x}_{n/2}$, which alternates ± 1.

Proof. We will first see that $\boldsymbol{x}_1$ and $\boldsymbol{y}_1$ are eigenvectors by drawing the ring graph on the unit circle in the natural way: plot vertex a at point $(\cos(2\pi a/n), \sin(2\pi a/n))$.

You can see that the average of the neighbors of a vertex is a vector pointing in the same direction as the vector associated with that vertex. This should make it obvious that both the x and y coordinates in this figure are eigenvectors of the same eigenvalue. The same holds for all k.

Alternatively, we can verify that these are eigenvectors by a simple computation.



(a) The ring graph on 9 vertices.

(b) The eigenvectors for $k = 2$.

Figure 6.2:

$$
\begin{aligned}
\left(L_{R_n} \boldsymbol{x}_k\right)(a) &= 2\boldsymbol{x}_k(a) - \boldsymbol{x}_k(a+1) - \boldsymbol{x}_k(a-1) \\
&= 2\cos(2\pi ka/n) - \cos(2\pi k(a+1)/n) - \cos(2\pi k(a-1)/n) \\
&= 2\cos(2\pi ka/n) - \cos(2\pi ka/n)\cos(2\pi k/n) + \sin(2\pi ka/n)\sin(2\pi k/n) \\
&\qquad\qquad - \cos(2\pi ka/n)\cos(2\pi k/n) - \sin(2\pi ka/n)\sin(2\pi k/n) \\
&= 2\cos(2\pi ka/n) - \cos(2\pi ka/n)\cos(2\pi k/n) - \cos(2\pi ka/n)\cos(2\pi k/n) \\
&= (2 - 2\cos(2\pi k/n))\cos(2\pi ka/n) \\
&= (2 - \cos(2\pi k/n))\boldsymbol{x}_k(a).
\end{aligned}
$$

The computation for $\boldsymbol{y}_k$ follows similarly. □

6.6 The Path Graph

We will derive the eigenvalues and eigenvectors of the path graph from those of the ring graph. To begin, I will number the vertices of the ring a little differently, as in Figure 6.3.

Lemma 6.6.1. *Let $P_n = (V, E)$ where $V = \{1, \ldots, n\}$ and $E = \{(a, a+1) : 1 \leq a < n\}$. The Laplacian of P_n has the same eigenvalues as R_{2n}, excluding 2. That is, P_n has eigenvalues namely $2(1 - \cos(\pi k/n))$, and eigenvectors*

$$\boldsymbol{v}_k(a) = \cos(\pi ka/n - \pi k/2n).$$

for $0 \leq k < n$

Proof. We derive the eigenvectors and eigenvalues by treating P_n as a quotient of R_{2n}: we will identify vertex a of P_n with vertices a and $a+n$ of R_{2n} (under the new numbering of R_{2n}). These are pairs of vertices that are above each other in the figure that I drew.

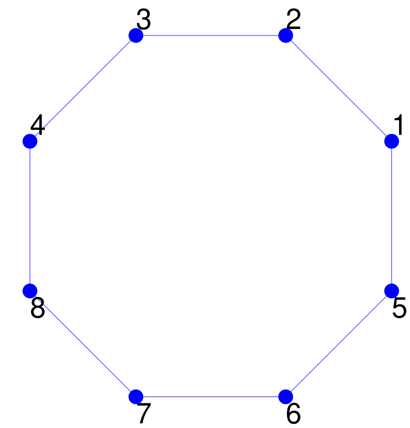


Figure 6.3: The ring on 8 vertices, numbered differently

Let $\boldsymbol{I}_n$ be the n-dimensional identity matrix. You should check that

$$\begin{pmatrix} \boldsymbol{I}_n & \boldsymbol{I}_n \end{pmatrix} \boldsymbol{L}_{R_{2n}} \begin{pmatrix} \boldsymbol{I}_n \\ \boldsymbol{I}_n \end{pmatrix} = 2\boldsymbol{L}_{P_n}.$$

If there is an eigenvector $\boldsymbol{\psi}$ of R_{2n} with eigenvalue λ for which $\boldsymbol{\psi}(a) = \boldsymbol{\psi}(a+n)$ for $1 \leq a \leq n$, then the above equation gives us a way to turn this into an eigenvector of P_n: Let $\boldsymbol{\phi} \in \mathbb{R}^n$ be the vector for which

$$\boldsymbol{\phi}(a) = \boldsymbol{\psi}(a), \text{for } 1 \leq a \leq n.$$

Then,

$$\begin{pmatrix} \boldsymbol{I}_n \\ \boldsymbol{I}_n \end{pmatrix} \boldsymbol{\phi} = \boldsymbol{\psi}, \quad \boldsymbol{L}_{R_{2n}} \begin{pmatrix} \boldsymbol{I}_n \\ \boldsymbol{I}_n \end{pmatrix} \boldsymbol{\phi} = \lambda \boldsymbol{\psi}, \quad \text{and} \quad \begin{pmatrix} \boldsymbol{I}_n & \boldsymbol{I}_n \end{pmatrix} \boldsymbol{L}_{R_{2n}} \begin{pmatrix} \boldsymbol{I}_n \\ \boldsymbol{I}_n \end{pmatrix} \boldsymbol{\psi} = 2\lambda \boldsymbol{\phi}.$$

So, if we can find such a vector $\boldsymbol{\psi}$, then the corresponding $\boldsymbol{\phi}$ is an eigenvector of P_n of eigenvalue λ.

As you've probably guessed, we can find such vectors $\boldsymbol{\psi}$. I've drawn one in Figure 6.3. For each of the two-dimensional eigenspaces of R_{2n}, we get one such a vector. These provide eigenvectors of eigenvalue

$$2(1 - \cos(\pi k/n)),$$

for $1 \leq k < n$. Thus, we now know $n-1$ distinct eigenvalues. The last, of course, is zero. □

The type of quotient used in the above argument is known as an *equitable partition*. You can find a extensive exposition of these in Godsil's book [God93].