The second form and curvature

We now Know:

$$S_p : T_pM \to T_pM, \quad S_p = -D\vec{g}$$

(where $\vec{g} : M \to S^2$ is the Gauss map taking each point $\vec{p} \in M$ to the normal vector $\vec{n}$ at $\vec{p}$.)

$$S_p(\vec{v}) = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} \ell & m \\ m & n \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

if $\vec{v} = a\vec{x}_u + b\vec{x}_v$, and

$$\ell = -\langle \vec{n}_u, \vec{x}_u \rangle$$
$$m = -\langle \vec{n}_u, \vec{x}_v \rangle$$
$$n = -\langle \vec{n}_v, \vec{x}_v \rangle$$

$$\langle \vec{v}, S_p(\vec{v}) \rangle_{I_p} = \langle \vec{v}, \begin{bmatrix} \ell & m \\ m & n \end{bmatrix} \vec{v} \rangle_{\mathbb{R}^2}$$

$= \langle \vec{v}, \vec{v} \rangle_{II_p} =$ signed curvature of $\vec{v}, \vec{n}$ slice curve.

$S_p(\vec{v})$ is self-adjoint wrt I_p,
so $\langle \vec{v}, S_p(\vec{w}) \rangle_{I_p} = \langle S_p(\vec{v}), \vec{w} \rangle_{I_p}$.

$S_p(\vec{v})$ has two orthogonal, unit eigenvectors $\vec{v}_1, \vec{v}_2$ with real eigenvalues $\kappa_1 \geq \kappa_2$.

If $\vec{v} = \cos\theta\, \vec{v}_1 + \sin\theta\, \vec{v}_2$, then

$$\langle \vec{v}, \vec{v} \rangle_{II_p} = \kappa_1 \cos\theta\, \vec{v}_1 + \kappa_2 \sin\theta\, \vec{v}_2$$

Goal: Show how to use this to compute and understand.

Proposition. We may compute II_p using the alternate formulas

$$\ell = \langle \vec{n}, \vec{X}_{uu} \rangle_{\mathbb{R}^3} \quad m = \langle \vec{n}, \vec{X}_{uv} \rangle_{\mathbb{R}^3} \quad n = \langle \vec{n}, \vec{X}_{vv} \rangle_{\mathbb{R}^3}$$

Proof. We have already shown

$$m = \langle -\vec{n}_u, \vec{X}_v \rangle = \langle \vec{n}, \vec{X}_{uv} \rangle = \langle \vec{n}, \vec{X}_{vu} \rangle = \langle -\vec{n}_v, \vec{X}_u \rangle$$

by taking $\frac{\partial}{\partial v}$ of $\langle \vec{n}, \vec{X}_u \rangle \equiv 0$.

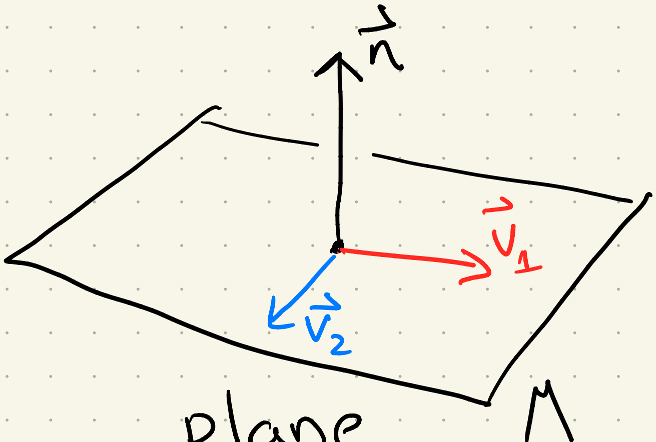
Similarly,

$$0 = \frac{\partial}{\partial u} \langle \vec{n}, \vec{X}_u \rangle = \langle \vec{n}_u, \vec{X}_u \rangle + \langle \vec{n}, \vec{X}_{uu} \rangle$$

$$0 = \frac{\partial}{\partial v} \langle \vec{n}, \vec{X}_v \rangle = \langle \vec{n}_v, \vec{X}_v \rangle + \langle \vec{n}, \vec{X}_{vv} \rangle.$$

$\blacksquare$

Examples.



plane

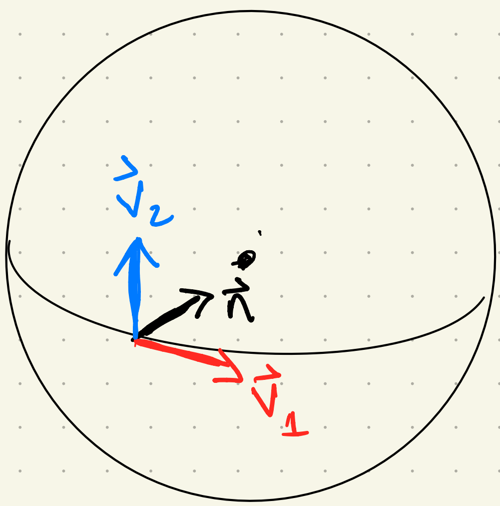
$\vec{n}$ is constant

$D\vec{g} \equiv 0$

$S_{\vec{p}} \equiv 0$

$\kappa_1 = \kappa_2 = 0$

$\vec{v}_1, \vec{v}_2$ arbitrary



sphere of radius a
(oriented <u>in</u>wards)

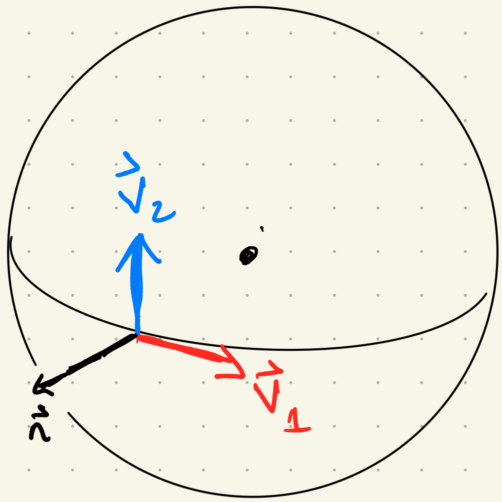
$\vec{n} = -\frac{1}{a}\vec{x}$

$D\vec{g} \equiv -\frac{1}{a} I$

$S_{\vec{p}} \equiv +\frac{1}{a} I$

$\kappa_1 = \kappa_2 = \frac{1}{a}$

(slice curves bending <u>towards</u> normal)

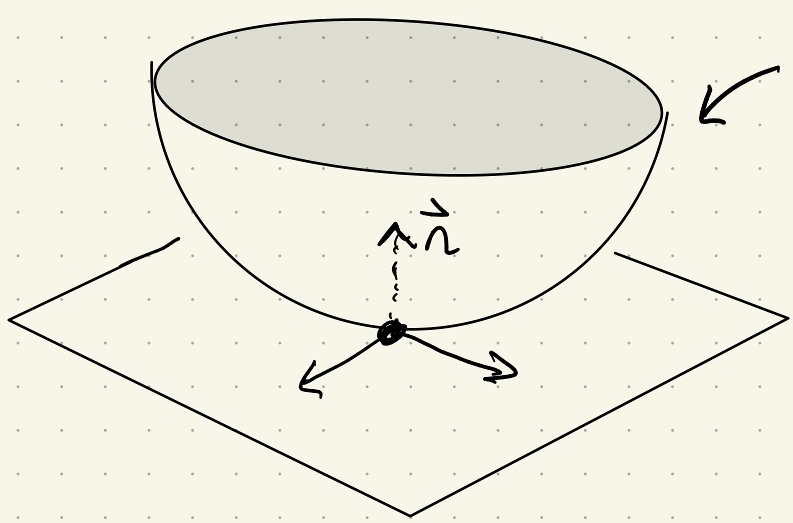


sphere of radius a
(oriented <u>outwards</u>)

$$\vec{n} = \frac{1}{a}\vec{x}$$
$$D\vec{g} \equiv \frac{1}{a} I$$
$$S_{\vec{p}} \equiv -\frac{1}{a} I$$
$$\kappa_1 = \kappa_2 = -\frac{1}{a}$$

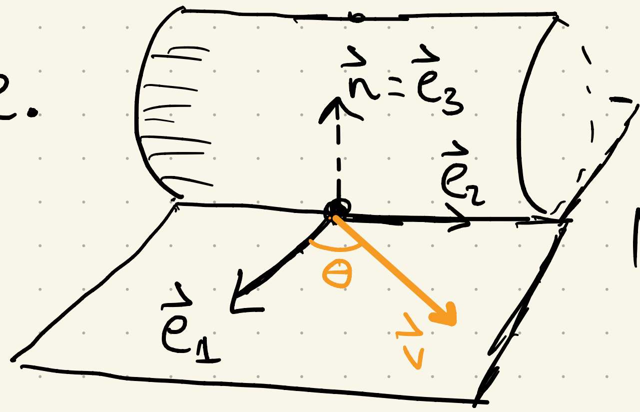
(slice curves bending <u>away from</u> normal)

If the signs bother you, recall that the "positive" parametrization of S^2 as a graph has <u>inward</u> normal.



$$z = 1 - \sqrt{1 - x^2 - y^2}$$

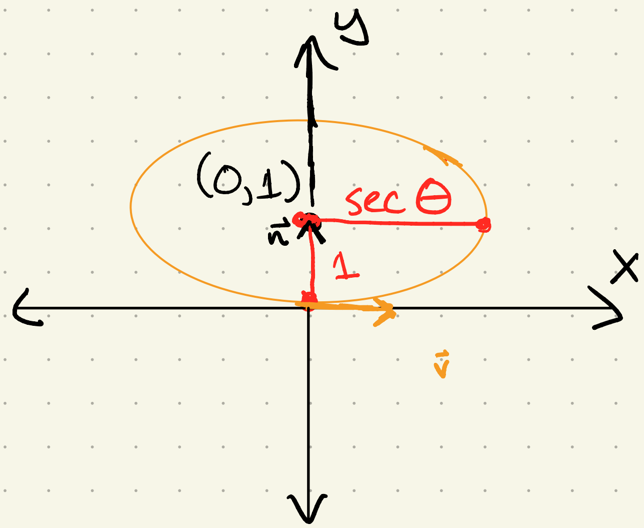
Example.

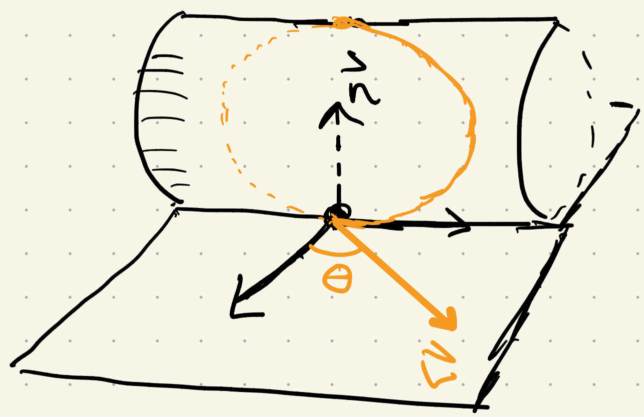


$M = \{ x_1^2 + x_3^2 = 1 \}$

At $\vec{p} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$, $\vec{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$

so the slice plane is parametrized by $(x, y) \mapsto x\vec{v} + y\vec{n} + \vec{p}$





The slice curve satisfies $x^2 + z^2 = 1$ which is an ellipse (unless $\theta = \pi/2$).

We prove it by computing

$$X_1 = x\cos\theta \qquad X_3 = y-1$$

so the equation $X_1^2 + X_3^2 = 1$ becomes

$$1 = \cos^2\theta\, x^2 + (y-1)^2$$
$$= \frac{(x-0)^2}{\sec^2\theta} + \frac{(y-1)^2}{1}.$$

We computed long ago that the signed curvature of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ was

$$\kappa_\pm(t) = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}$$

in the ccw parametrization

$$\vec{\alpha}(t) = \begin{bmatrix} a\cos t + x_0 \\ b\sin t + y_0 \end{bmatrix}$$

since we are evaluating at $t = -\pi/2$, with $a = \sec\theta$, $b = 1$, we get

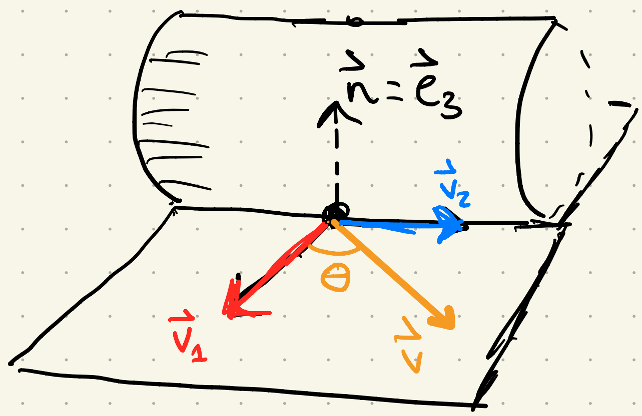
$$K_{\pm}(\theta) = \frac{\sec\theta}{\sec^3\theta} = \cos^2\theta$$

Suppose that $\vec{v}_1, \vec{v}_2$ are principal directions. We know that if $\vec{v} = \cos\theta\,\vec{v}_1 + \sin\theta\,\vec{v}_2$ we have

$$K_{\pm}(\theta) = K_1 \cos^2\theta + K_2 \sin^2\theta$$

We conclude that

$$K_1 = 1, \quad \vec{v}_1 = \vec{e}_1$$
$$K_2 = 0, \quad \vec{v}_2 = \vec{e}_2$$



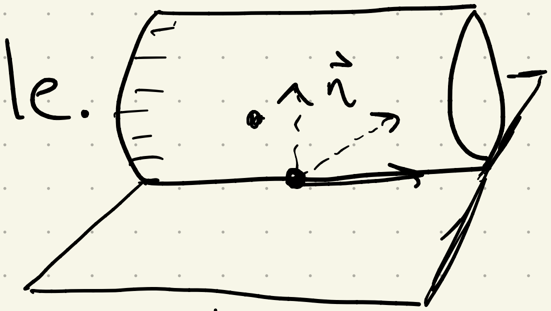
Note: We can just recompute

$$\vec{\alpha}'(\varphi) = \begin{bmatrix} -a\sin\varphi \\ b\cos\varphi \end{bmatrix} \qquad \vec{\alpha}''(\varphi) = \begin{bmatrix} -a\cos\varphi \\ -b\sin\varphi \end{bmatrix}$$

$$K_{\pm}(\varphi) = \frac{\langle \vec{\alpha}''(\varphi), \vec{\alpha}'(\varphi)^{\perp} \rangle}{\langle \vec{\alpha}'(\varphi), \vec{\alpha}'(\varphi) \rangle^{3/2}}$$

$$= \frac{\left\langle \begin{bmatrix} -a\cos\varphi \\ -b\sin\varphi \end{bmatrix}, \begin{bmatrix} -b\cos\varphi \\ -a\sin\varphi \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} -a\sin\varphi \\ b\cos\varphi \end{bmatrix}, \begin{bmatrix} -a\sin\varphi \\ b\cos\varphi \end{bmatrix} \right\rangle^{3/2}}$$

$$= \frac{ab}{(a^2\sin^2\varphi + b^2\cos\varphi)^{3/2}}$$

Example.  $\vec{x}(u,v) = \begin{bmatrix} \cos u \\ v \\ \sin u \end{bmatrix}$

We compute

$$\vec{X}_u = (-\sin u, 0, \cos u)$$

$$\vec{X}_v = (0, 1, 0)$$

$$\vec{n} = (-\cos u, 0, -\sin u)$$

$$\vec{X}_{uu} = (-\cos u, 0, -\sin u)$$

$$\vec{X}_{uv} = (0, 0, 0)$$

$$\vec{X}_{vv} = (0, 0, 0)$$

so

$$E = \langle \vec{X}_u, \vec{X}_u \rangle = 1 \qquad \ell = \langle \vec{X}_{uu}, \vec{n} \rangle = 1$$

$$F = \langle \vec{X}_u, \vec{X}_v \rangle = 0 \qquad m = \langle \vec{X}_{uv}, \vec{n} \rangle = 0$$

$$G = \langle \vec{X}_v, \vec{X}_v \rangle = 1 \qquad n = \langle \vec{X}_{vv}, \vec{n} \rangle = 0$$

Thus, at $u = -\pi/2$, $v = 0$, we have

$$S_p = I_p^{-1} II_p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

It follows that the eigenvalues K_1, K_2 are $1, 0$ with eigenvectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{x}_u + 0\vec{x}_v = (1,0,0) = \vec{e}_1$$

in the $\vec{x}_u, \vec{x}_v$ basis

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0\vec{x}_u + \vec{x}_v = (0,1,0) = \vec{e}_2$$

which agrees with our previous calculation.

We now introduce

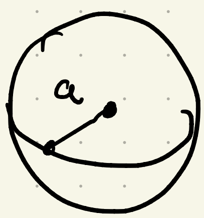
Definition. The Gauss curvature

$$K = \det S_p = \kappa_1 \kappa_2$$

and the mean curvature

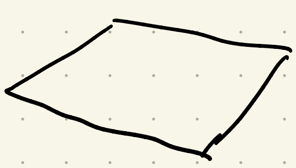
$$H = \frac{1}{2} \operatorname{tr} S_p = \frac{1}{2}(\kappa_1 + \kappa_2)$$

Examples.

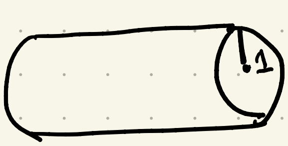


$K = \frac{1}{a^2}$, $H = \frac{1}{a}$ (inward normal)

$= -\frac{1}{a}$ (outward normal)



$K = 0$, $H = 0$



$K = 0$, $H = \frac{1}{2}$ (inward normal)

We note that

<u>Proposition</u>: If $K<0$, there are exactly two directions with $\langle \vec{v}, \vec{v} \rangle_{\mathrm{II}_p} = 0$. If $K>0$, there are <u>no</u> directions with $\langle \vec{v}, \vec{v} \rangle_{\mathrm{II}_p} = 0$.

<u>Proof</u>. If $\vec{v} = \cos\theta \, \vec{v}_1 + \sin\theta \, \vec{v}_2$,

$$\langle \vec{v}, \vec{v} \rangle = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta$$

so $\langle \vec{v}, \vec{v} \rangle = 0 \iff$

$$\tan^2\theta = -\frac{\kappa_1}{\kappa_2}$$

or

$$\theta = \arctan\left(\pm\sqrt{-\kappa_1/\kappa_2} \right).$$

There are exactly two such θ in $[-\pi/2, \pi/2] \iff -\kappa_1/\kappa_2 > 0$. Notice that $\tan(\theta+\pi) = \tan\theta$, so there are also two such θ in $[\pi/2, 3\pi/2]$. $\blacksquare$

<u>Definition</u>. If $\langle \vec{v}, \vec{v} \rangle_{\mathrm{II}_p} = 0$, we say $\vec{v}$ points in an asymptotic direction.