CHAPTER 1

Curves

1. Examples, Arclength Parametrization

We say a vector function $\mathbf{f}\colon (a,b) \to \mathbb{R}^3$ is $\mathcal{C}^k$ $(k = 0, 1, 2, \dots)$ if $\mathbf{f}$ and its first k derivatives, $\mathbf{f}', \mathbf{f}'', \dots, \mathbf{f}^{(k)}$, exist and are all continuous. We say $\mathbf{f}$ is *smooth* if $\mathbf{f}$ is $\mathcal{C}^k$ for every positive integer k. A *parametrized curve* is a $\mathcal{C}^3$ (or smooth) map $\boldsymbol{\alpha}\colon I \to \mathbb{R}^3$ for some interval $I = (a,b)$ or $[a,b]$ in $\mathbb{R}$ (possibly infinite). We say $\boldsymbol{\alpha}$ is *regular* if $\boldsymbol{\alpha}'(t) \neq \mathbf{0}$ for all $t \in I$.

We can imagine a particle moving along the path $\boldsymbol{\alpha}$, with its position at time t given by $\boldsymbol{\alpha}(t)$. As we learned in vector calculus,

$$\boldsymbol{\alpha}'(t) = \frac{d\boldsymbol{\alpha}}{dt} = \lim_{h\to 0} \frac{\boldsymbol{\alpha}(t+h) - \boldsymbol{\alpha}(t)}{h}$$

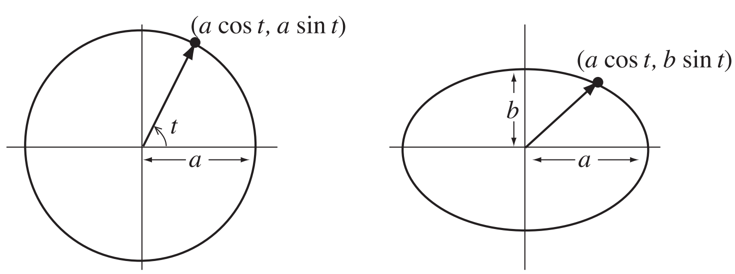
is the *velocity* of the particle at time t. The velocity vector $\boldsymbol{\alpha}'(t)$ is tangent to the curve at $\boldsymbol{\alpha}(t)$ and its length, $\|\boldsymbol{\alpha}'(t)\|$, is the speed of the particle.

Example 1. We begin with some standard examples.

(a) Familiar from linear algebra and vector calculus is a parametrized line: Given points P and Q in $\mathbb{R}^3$, we let $\mathbf{v} = \overrightarrow{PQ} = Q - P$ and set $\boldsymbol{\alpha}(t) = P + t\mathbf{v}$, $t \in \mathbb{R}$. Note that $\boldsymbol{\alpha}(0) = P$, $\boldsymbol{\alpha}(1) = Q$, and for $0 \le t \le 1$, $\boldsymbol{\alpha}(t)$ is on the line segment $\overline{PQ}$. We ask the reader to check in Exercise 8 that of all paths from P to Q, the "straight line path" $\boldsymbol{\alpha}$ gives the shortest. This is typical of problems we shall consider in the future.

(b) Essentially by the very definition of the trigonometric functions cos and sin, we obtain a very natural parametrization of a circle of radius a, as pictured in Figure 1.1(a):

$$\boldsymbol{\alpha}(t) = a\big(\cos t, \sin t\big) = \big(a\cos t, a\sin t\big), \quad 0 \le t \le 2\pi.$$



(a) (b)

FIGURE 1.1

(c) Now, if $a, b > 0$ and we apply the linear map

$$T: \mathbb{R}^2 \to \mathbb{R}^2, \quad T(x, y) = (ax, by),$$

we see that the unit circle $x^2+y^2 = 1$ maps to the ellipse $x^2/a^2+y^2/b^2 = 1$. Since $T(\cos t, \sin t) = (a \cos t, b \sin t)$, the latter gives a natural parametrization of the ellipse, as shown in Figure 1.1(b).

(d) Consider the two cubic curves in $\mathbb{R}^2$ illustrated in Figure 1.2. On the left is the *cuspidal cubic*

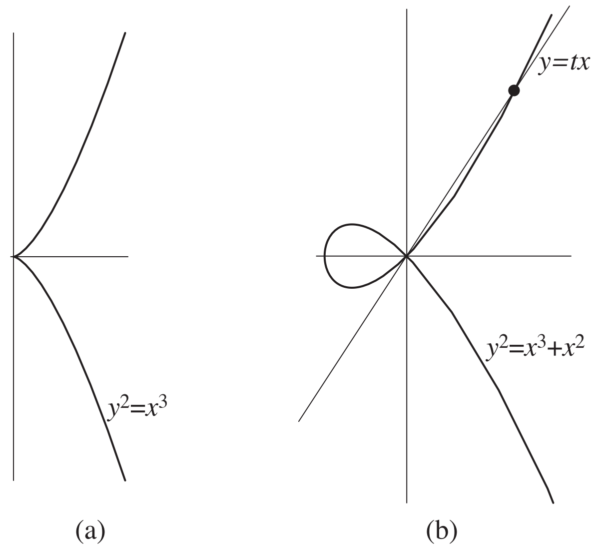


FIGURE 1.2

$y^2 = x^3$, and on the right is the *nodal cubic* $y^2 = x^3+x^2$. These can be parametrized, respectively, by the functions

$$\boldsymbol{\alpha}(t) = (t^2, t^3) \qquad \text{and} \qquad \boldsymbol{\alpha}(t) = (t^2 - 1, t(t^2 - 1)).$$

(In the latter case, as the figure suggests, we see that the line $y = tx$ intersects the curve when $(tx)^2 = x^2(x + 1)$, so $x = 0$ or $x = t^2 - 1$.)

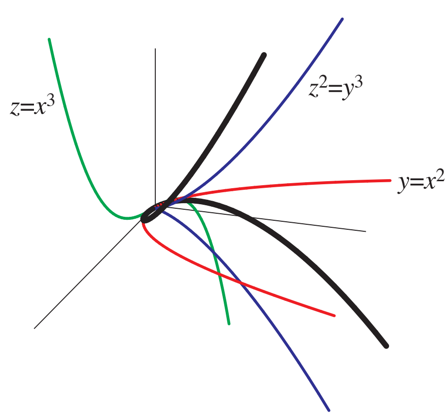


FIGURE 1.3

(e) Now consider the *twisted cubic* in $\mathbb{R}^3$, illustrated in Figure 1.3, given by

$$\boldsymbol{\alpha}(t) = (t, t^2, t^3), \quad t \in \mathbb{R}.$$

Its projections in the xy-, xz-, and yz-coordinate planes are, respectively, $y = x^2$, $z = x^3$, and $z^2 = y^3$ (the cuspidal cubic).

(f) Our next example is a classic called the *cycloid*: It is the trajectory of a dot on a rolling wheel (circle). Consider the illustration in Figure 1.4. Assuming the wheel rolls without slipping, the

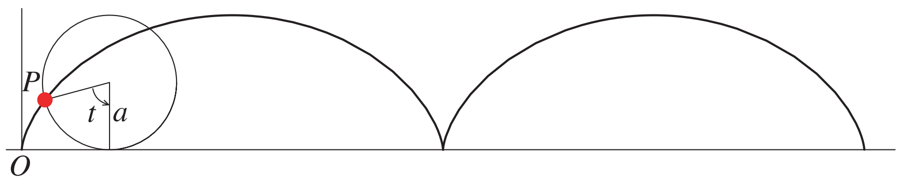


FIGURE 1.4

distance it travels along the ground is equal to the length of the circular arc subtended by the angle through which it has turned. That is, if the radius of the circle is a and it has turned through angle t, then the point of contact with the x-axis, Q, is at units to the right. The vector from the origin to

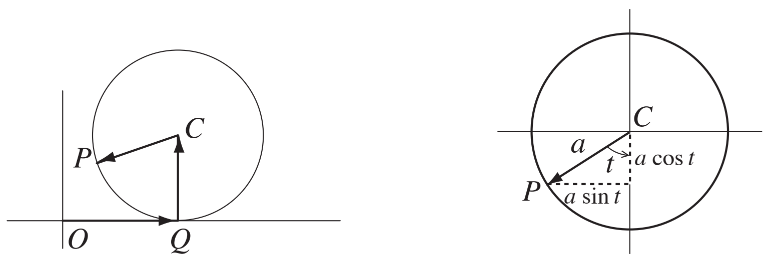


FIGURE 1.5

the point P can be expressed as the sum of the three vectors $\overrightarrow{OQ}$, $\overrightarrow{QC}$, and $\overrightarrow{CP}$ (see Figure 1.5):

$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OQ} + \overrightarrow{QC} + \overrightarrow{CP} \\ &= (at, 0) + (0, a) + (-a\sin t, -a\cos t),\end{aligned}$$

and hence the function

$$\boldsymbol{\alpha}(t) = (at - a\sin t, a - a\cos t) = a(t - \sin t, 1 - \cos t), \quad t \in \mathbb{R}$$

gives a parametrization of the cycloid.

(g) A (circular) *helix* is the screw-like path of a bug as it walks uphill on a right circular cylinder at a constant slope or pitch. If the cylinder has radius a and the slope is b/a, we can imagine drawing a line of that slope on a piece of paper $2\pi a$ units long, and then rolling the paper up into a cylinder. The line gives one revolution of the helix, as we can see in Figure 1.6. If we take the axis of the cylinder to be vertical, the projection of the helix in the horizontal plane is a circle of radius a, and so we obtain the parametrization $\boldsymbol{\alpha}(t) = (a\cos t, a\sin t, bt)$.

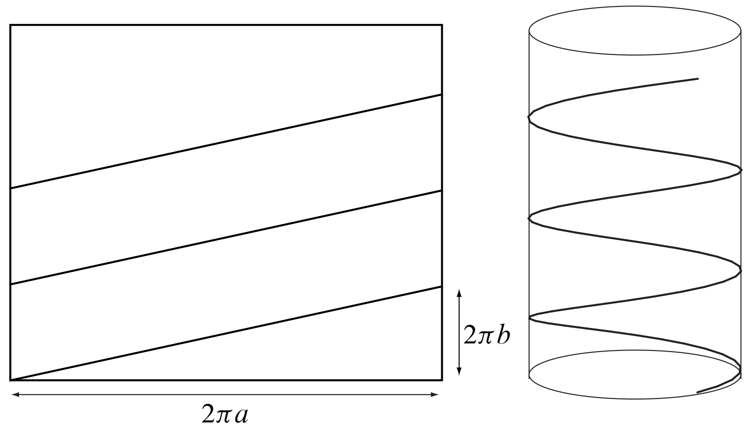


FIGURE 1.6

Brief review of hyperbolic trigonometric functions. Just as the circle $x^2+y^2=1$ is parametrized by $(\cos\theta, \sin\theta)$, the portion of the hyperbola $x^2-y^2=1$ lying to the right of the y-axis, as shown in Figure 1.7, is parametrized by $(\cosh t, \sinh t)$, where

$$\cosh t = \frac{e^t + e^{-t}}{2} \qquad \text{and} \qquad \sinh t = \frac{e^t - e^{-t}}{2}.$$

By analogy with circular trigonometry, we set $\tanh t = \dfrac{\sinh t}{\cosh t}$ and $\operatorname{sech} t = \dfrac{1}{\cosh t}$. The following

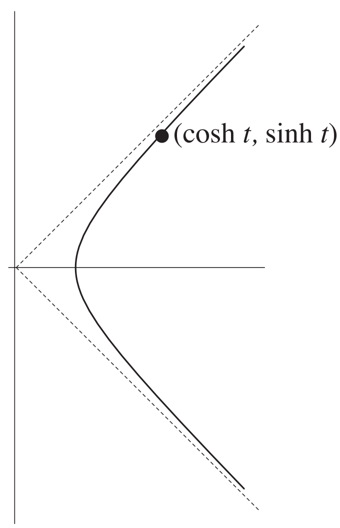


FIGURE 1.7

formulas are easy to check:

$$\cosh^2 t - \sinh^2 t = 1, \qquad \tanh^2 t + \operatorname{sech}^2 t = 1$$

$$\sinh'(t) = \cosh t, \qquad \cosh'(t) = \sinh t, \qquad \tanh'(t) = \operatorname{sech}^2 t, \qquad \operatorname{sech}'(t) = -\tanh t \ \operatorname{sech} t.$$

(h) When a uniform and flexible chain hangs from two pegs, its weight is uniformly distributed along its length. The shape it takes is called a *catenary*.[1] As we ask the reader to check in Exercise 9, the catenary is the graph of $f(x) = C\cosh(x/C)$, for any constant $C > 0$. This curve will appear

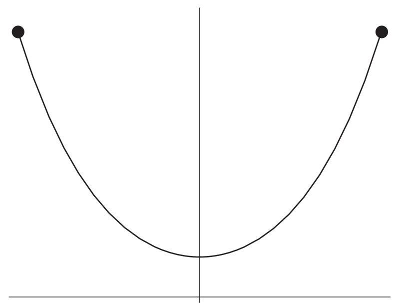


FIGURE 1.8

numerous times in this course. ∇

Example 2. One of the more interesting curves that arise "in nature" is the *tractrix*.[2] The traditional story is this: A dog is at the end of a 1-unit leash and buries a bone at $(0,1)$ as his owner begins to walk down the x-axis, starting at the origin. The dog tries to get back to the bone, so he always pulls the leash taut as he is dragged along the tractrix by his owner. His pulling the leash taut means that the leash will be tangent to the curve. When the master is at $(t,0)$, let the dog's position be $(x(t), y(t))$, and let the leash

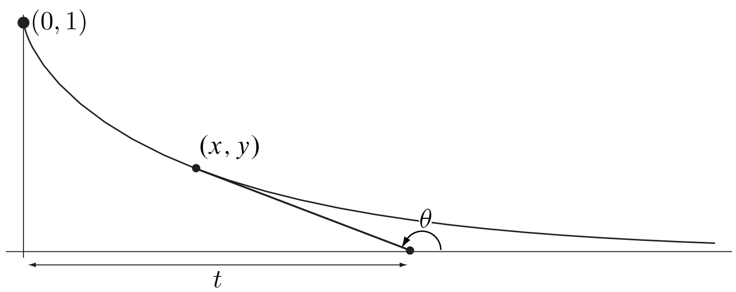


FIGURE 1.9

make angle $\theta(t)$ with the positive x-axis. Then we have $x(t) = t + \cos\theta(t)$, $y(t) = \sin\theta(t)$, so

$$\tan\theta(t) = \frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\cos\theta(t)\theta'(t)}{1-\sin\theta(t)\theta'(t)}.$$

Therefore, $\theta'(t) = \sin\theta(t)$. Separating variables and integrating, we have $\int d\theta/\sin\theta = \int dt$, and so $t = -\ln(\csc\theta + \cot\theta) + c$ for some constant c. Since $\theta = \pi/2$ when $t = 0$, we see that $c = 0$. Now, since $\csc\theta + \cot\theta = \dfrac{1+\cos\theta}{\sin\theta} = \dfrac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} = \cot(\theta/2)$, we can rewrite this as $t = \ln\tan(\theta/2)$. Thus, we can parametrize the tractrix by

$$\boldsymbol{\alpha}(\theta) = \big(\cos\theta + \ln\tan(\theta/2), \sin\theta\big), \quad \pi/2 \le \theta < \pi.$$

[1] From the Latin *catēna*, chain.

[2] From the Latin *trahere, tractus*, to pull.

Alternatively, since $\tan(\theta/2) = e^t$, we have

$$\sin\theta = 2\sin(\theta/2)\cos(\theta/2) = \frac{2e^t}{1+e^{2t}} = \frac{2}{e^t + e^{-t}} = \operatorname{sech} t$$
$$\cos\theta = \cos^2(\theta/2) - \sin^2(\theta/2) = \frac{1-e^{2t}}{1+e^{2t}} = \frac{e^{-t} - e^t}{e^t + e^{-t}} = -\tanh t,$$

and so we can parametrize the tractrix instead by

$$\boldsymbol{\beta}(t) = \big(t - \tanh t, \operatorname{sech} t\big), \quad t \geq 0. \qquad \nabla$$

The fundamental concept underlying the geometry of curves is the arclength of a parametrized curve.

Definition. If $\boldsymbol{\alpha}\colon [a,b] \to \mathbb{R}^3$ is a parametrized curve, then for any $a \leq t \leq b$, we define its *arclength* from a to t to be $s(t) = \displaystyle\int_a^t \|\boldsymbol{\alpha}'(u)\|\,du$. That is, the distance a particle travels—the arclength of its trajectory—is the integral of its speed.

An alternative approach is to start with the following

Definition. Let $\boldsymbol{\alpha}\colon [a,b] \to \mathbb{R}^3$ be a (continuous) parametrized curve. Given a partition $\mathcal{P} = \{a = t_0 < t_1 < \cdots < t_k = b\}$ of the interval $[a,b]$, let

$$\ell(\boldsymbol{\alpha}, \mathcal{P}) = \sum_{i=1}^{k} \|\boldsymbol{\alpha}(t_i) - \boldsymbol{\alpha}(t_{i-1})\|.$$

That is, $\ell(\boldsymbol{\alpha}, \mathcal{P})$ is the length of the inscribed polygon with vertices at $\boldsymbol{\alpha}(t_i)$, $i = 0, \ldots, k$, as indicated in

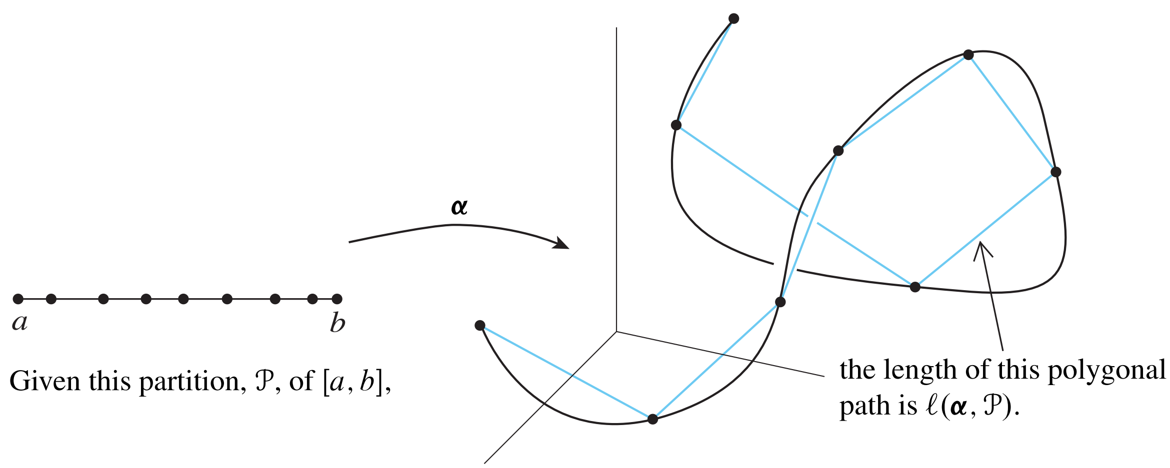


FIGURE 1.10

Figure 1.10. We define the *arclength* of $\boldsymbol{\alpha}$ to be

$$\operatorname{length}(\boldsymbol{\alpha}) = \sup\{\ell(\boldsymbol{\alpha}, \mathcal{P}) : \mathcal{P} \text{ a partition of } [a,b]\},$$

provided the set of polygonal lengths is bounded above.

Now, using this definition, we can *prove* that the distance a particle travels is the integral of its speed. We will need to use the result of Exercise A.2.4.