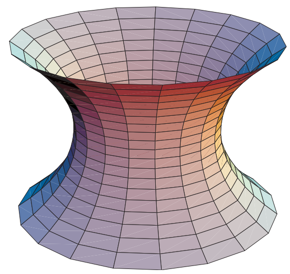

FIGURE 1.9

d. (For extra credit) Show that as $R \to \infty$, the area of the inner catenoid is asymptotic to $2\pi R^2$ and the area of the outer catenoid is asymptotic to $4\pi R$.

17. There are two obvious families of circles on a torus. Find a third family. (Hint: Look for a plane that is tangent to the torus at *two* points. Using the parametrization of the torus, you should be able to find equations (either parametric or cartesian) for the curve in which the bitangent plane intersects the torus.)

2. The Gauss Map and the Second Fundamental Form

Given a regular parametrized surface M, the function $\mathbf{n}: M \to \Sigma$ that assigns to each point $P \in M$ the unit normal $\mathbf{n}(P)$, as pictured in Figure 2.1, is called the *Gauss map* of M. As we shall see in this chapter,

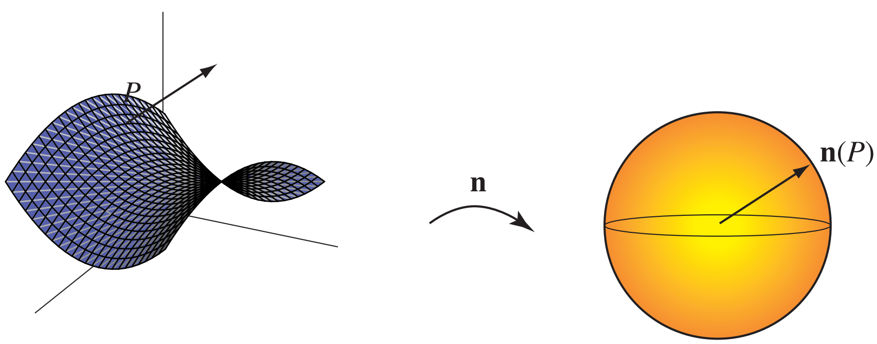

FIGURE 2.1

most of the geometric information about our surface M is encapsulated in the mapping $\mathbf{n}$.

Example 1. A few basic examples are these.

(a) On a plane, the tangent plane never changes, so the Gauss map is a constant.
(b) On a cylinder, the tangent plane is constant along the rulings, so the Gauss map sends the entire surface to an equator of the sphere.
(c) On a sphere centered at the origin, the Gauss map is merely the (normalized) position vector.
(d) On a saddle surface (as pictured in Figure 2.1), the Gauss map appears to "reverse orientation": As we move counterclockwise in a small circle around P, we see that the unit vector $\mathbf{n}$ turns clockwise around $\mathbf{n}(P)$. ∇

Recall from the Appendix that for any function f on M (scalar- or vector-valued) and any tangent vector $\mathbf{V} \in T_P M$, we can compute the directional derivative $D_{\mathbf{V}} f(P)$ by choosing a curve $\boldsymbol{\alpha}: (-\varepsilon, \varepsilon) \to M$ with $\boldsymbol{\alpha}(0) = P$ and $\boldsymbol{\alpha}'(0) = \mathbf{V}$ and computing $(f \circ \boldsymbol{\alpha})'(0)$.

To understand the shape of M at the point P, we might try to understand the curvature at P of various curves in M. Perhaps the most obvious thing to try is various *normal* slices of M. That is, we slice M with the plane through P spanned by $\mathbf{n}(P)$ and a *unit* vector $\mathbf{V} \in T_P M$. Various such normal slices are shown for a saddle surface in Figure 2.2. Let $\boldsymbol{\alpha}$ be the arclength-parametrized curve obtained by taking such

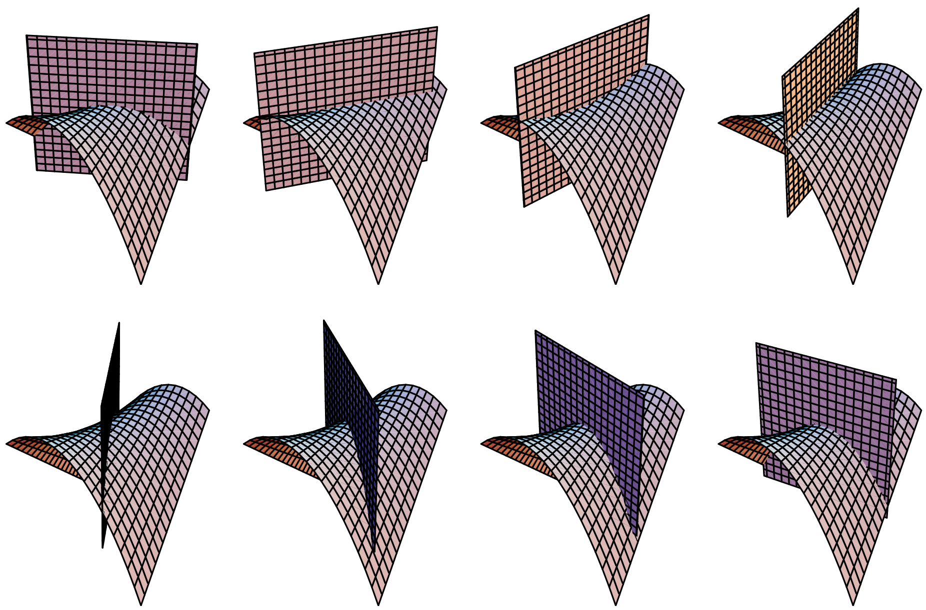

FIGURE 2.2

a normal slice. We have $\boldsymbol{\alpha}(0) = P$ and $\boldsymbol{\alpha}'(0) = \mathbf{V}$. Then since the curve lies in the plane spanned by $\mathbf{n}(P)$ and $\mathbf{V}$, the principal normal of the curve at P must be $\pm\mathbf{n}(P)$ ($+$ if the curve is curving towards $\mathbf{n}$, $-$ if it's curving away). Since $(\mathbf{n} \circ \boldsymbol{\alpha}(s)) \cdot \mathbf{T}(s) = 0$ for all s near 0, applying Lemma 2.1 of Chapter 1 yet again, we have:

$$(\dagger) \qquad \pm\kappa(P) = \kappa \mathbf{N} \cdot \mathbf{n}(P) = \mathbf{T}'(0) \cdot \mathbf{n}(P) = -\mathbf{T}(0) \cdot (\mathbf{n} \circ \boldsymbol{\alpha})'(0) = -D_{\mathbf{V}}\mathbf{n}(P) \cdot \mathbf{V}.$$

This leads us to study the directional derivative $D_{\mathbf{V}}\mathbf{n}(P)$ more carefully.

Proposition 2.1. *For any $\mathbf{V} \in T_P M$, the directional derivative $D_{\mathbf{V}}\mathbf{n}(P) \in T_P M$. Moreover, the linear map $S_P: T_P M \to T_P M$ defined by*

$$S_P(\mathbf{V}) = -D_{\mathbf{V}}\mathbf{n}(P)$$

is a symmetric *linear map; i.e., for any $\mathbf{U}, \mathbf{V} \in T_P M$, we have*

$$(*) \qquad S_P(\mathbf{U}) \cdot \mathbf{V} = \mathbf{U} \cdot S_P(\mathbf{V})$$

S_P is called the shape operator *at P.*

Proof. For any curve $\boldsymbol{\alpha}: (-\varepsilon, \varepsilon) \to M$ with $\boldsymbol{\alpha}(0) = P$ and $\boldsymbol{\alpha}'(0) = \mathbf{V}$, we observe that $\mathbf{n} \circ \boldsymbol{\alpha}$ has constant length 1. Thus, by Lemma 2.1 of Chapter 1, $D_{\mathbf{V}}\mathbf{n}(P) \cdot \mathbf{n}(P) = (\mathbf{n} \circ \boldsymbol{\alpha})'(0) \cdot (\mathbf{n} \circ \boldsymbol{\alpha})(0) = 0$, so $D_{\mathbf{V}}\mathbf{n}(P)$ is in the tangent plane to M at P. That S_P is a linear map is an immediate consequence of Proposition 2.3 of the Appendix.

Symmetry is our first important application of the equality of mixed partial derivatives. First we verify $(*)$ when $\mathbf{U} = \mathbf{x}_u$, $\mathbf{V} = \mathbf{x}_v$. Note that $\mathbf{n} \cdot \mathbf{x}_v = 0$, so $0 = \big(\mathbf{n} \cdot \mathbf{x}_v\big)_u = \mathbf{n}_u \cdot \mathbf{x}_v + \mathbf{n} \cdot \mathbf{x}_{vu}$. (Remember that we're writing $\mathbf{n}_u$ for $D_{\mathbf{x}_u}\mathbf{n}$.) Thus,

$$\begin{aligned} S_P(\mathbf{x}_u) \cdot \mathbf{x}_v = -D_{\mathbf{x}_u}\mathbf{n}(P) \cdot \mathbf{x}_v = -\mathbf{n}_u \cdot \mathbf{x}_v = \mathbf{n} \cdot \mathbf{x}_{vu} \\ = \mathbf{n} \cdot \mathbf{x}_{uv} = -\mathbf{n}_v \cdot \mathbf{x}_u = -D_{\mathbf{x}_v}\mathbf{n}(P) \cdot \mathbf{x}_u = S_P(\mathbf{x}_v) \cdot \mathbf{x}_u. \end{aligned}$$

Next, knowing this, we just write out general vectors $\mathbf{U}$ and $\mathbf{V}$ as linear combinations of $\mathbf{x}_u$ and $\mathbf{x}_v$: If $\mathbf{U} = a\mathbf{x}_u + b\mathbf{x}_v$ and $\mathbf{V} = c\mathbf{x}_u + d\mathbf{x}_v$, then

$$\begin{aligned} S_P(\mathbf{U}) \cdot \mathbf{V} &= S_P(a\mathbf{x}_u + b\mathbf{x}_v) \cdot (c\mathbf{x}_u + d\mathbf{x}_v) \\ &= \big(aS_P(\mathbf{x}_u) + bS_P(\mathbf{x}_v)\big) \cdot (c\mathbf{x}_u + d\mathbf{x}_v) \\ &= acS_P(\mathbf{x}_u) \cdot \mathbf{x}_u + adS_P(\mathbf{x}_u) \cdot \mathbf{x}_v + bcS_P(\mathbf{x}_v) \cdot \mathbf{x}_u + bdS_P(\mathbf{x}_v) \cdot \mathbf{x}_v \\ &= acS_P(\mathbf{x}_u) \cdot \mathbf{x}_u + adS_P(\mathbf{x}_v) \cdot \mathbf{x}_u + bcS_P(\mathbf{x}_u) \cdot \mathbf{x}_v + bdS_P(\mathbf{x}_v) \cdot \mathbf{x}_v \\ &= (a\mathbf{x}_u + b\mathbf{x}_v) \cdot \big(cS_P(\mathbf{x}_u) + dS_P(\mathbf{x}_v)\big) = \mathbf{U} \cdot S_P(\mathbf{V}), \end{aligned}$$

as required. $\square$

Proposition 2.2. *If the shape operator S_P is O for all $P \in M$, then M is a subset of a plane.*

Proof. Since the directional derivative of the unit normal $\mathbf{n}$ is $\mathbf{0}$ in every direction at every point P, we have $\mathbf{n}_u = \mathbf{n}_v = \mathbf{0}$ for any (local) parametrization $\mathbf{x}(u, v)$ of M. By Proposition 2.4 of the Appendix, it follows that $\mathbf{n}$ is constant. (This is why we assume our surfaces are connected.) $\square$

Example 2. Let M be a sphere of radius a centered at the origin. Then $\mathbf{n} = \dfrac{1}{a}\mathbf{x}(u, v)$, so for any P, we have $S_P(\mathbf{x}_u) = -\mathbf{n}_u = -\dfrac{1}{a}\mathbf{x}_u$ and $S_P(\mathbf{x}_v) = -\mathbf{n}_v = -\dfrac{1}{a}\mathbf{x}_v$, so S_P is $-1/a$ times the identity map on the tangent plane $T_P M$. ∇

It does not seem an easy task to give the matrix of the shape operator with respect to the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$. But, in general, the proof of Proposition 2.1 suggests that we define the second fundamental form, as follows. If $\mathbf{U}, \mathbf{V} \in T_P M$, we set

$$\mathrm{II}_P(\mathbf{U}, \mathbf{V}) = S_P(\mathbf{U}) \cdot \mathbf{V}.$$

Note that the formula (†) on p. 45 shows that the curvature of the normal slice in direction $\mathbf{V}$ (with $\|\mathbf{V}\| = 1$) is, in our new notation, given by

$$\pm\kappa = -D_{\mathbf{V}}\mathbf{n}(P) \cdot \mathbf{V} = S_P(\mathbf{V}) \cdot \mathbf{V} = \mathrm{II}_P(\mathbf{V}, \mathbf{V}).$$

As we did at the end of the previous section, we wish to give a matrix representation when we're working with a parametrized surface. As we saw in the proof of Proposition 2.1, we have

$$\begin{aligned} \ell &= \mathrm{II}_P(\mathbf{x}_u, \mathbf{x}_u) = -D_{\mathbf{x}_u}\mathbf{n} \cdot \mathbf{x}_u = \mathbf{x}_{uu} \cdot \mathbf{n} \\ m &= \mathrm{II}_P(\mathbf{x}_u, \mathbf{x}_v) = -D_{\mathbf{x}_u}\mathbf{n} \cdot \mathbf{x}_v = \mathbf{x}_{vu} \cdot \mathbf{n} = \mathbf{x}_{uv} \cdot \mathbf{n} = \mathrm{II}_P(\mathbf{x}_v, \mathbf{x}_u) \\ n &= \mathrm{II}_P(\mathbf{x}_v, \mathbf{x}_v) = -D_{\mathbf{x}_v}\mathbf{n} \cdot \mathbf{x}_v = \mathbf{x}_{vv} \cdot \mathbf{n}. \end{aligned}$$