(By the way, this explains the presence of the minus sign in the original definition of the shape operator.) We then write

$$\mathrm{II}_P = \begin{bmatrix} \ell & m \\ m & n \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{uu}\cdot\mathbf{n} & \mathbf{x}_{uv}\cdot\mathbf{n} \\ \mathbf{x}_{uv}\cdot\mathbf{n} & \mathbf{x}_{vv}\cdot\mathbf{n} \end{bmatrix}.$$

If, as before, $\mathbf{U} = a\mathbf{x}_u + b\mathbf{x}_v$ and $\mathbf{V} = c\mathbf{x}_u + d\mathbf{x}_v$, then

$$\begin{aligned} \mathrm{II}_P(\mathbf{U},\mathbf{V}) &= \mathrm{II}_P(a\mathbf{x}_u + b\mathbf{x}_v, c\mathbf{x}_u + d\mathbf{x}_v) \\ &= ac\,\mathrm{II}_P(\mathbf{x}_u,\mathbf{x}_u) + ad\,\mathrm{II}_P(\mathbf{x}_u,\mathbf{x}_v) + bc\,\mathrm{II}_P(\mathbf{x}_v,\mathbf{x}_u) + bd\,\mathrm{II}_P(\mathbf{x}_v,\mathbf{x}_v) \\ &= \ell(ac) + m(bc+ad) + n(bd). \end{aligned}$$

In the event that $\{\mathbf{x}_u, \mathbf{x}_v\}$ is an orthonormal basis for $T_P M$, we see that the matrix II_P represents the shape operator S_P. But it is not difficult to check (see Exercise 2) that, in general, the matrix of the linear map S_P with respect to the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ is given by

$$\mathrm{I}_P^{-1}\mathrm{II}_P = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} \ell & m \\ m & n \end{bmatrix}.$$

Remark. We proved in Proposition 2.1 that S_P is a symmetric linear map. This means that its matrix representation with respect to an orthonormal basis (or, more generally, orthogonal basis with vectors of equal length) *will* be symmetric: In this case the matrix I_P is a scalar multiple of the identity matrix and the matrix product remains symmetric.

By the Spectral Theorem, Theorem 1.3 of the Appendix, S_P has two real eigenvalues, traditionally denoted $k_1(P), k_2(P)$.

Definition. The eigenvalues of S_P are called the *principal curvatures* of M at P. Corresponding eigenvectors are called *principal directions*. A curve in M is called a *line of curvature* if its tangent vector at each point is a principal direction.

Recall that it also follows from the Spectral Theorem that the principal directions are orthogonal, so we can always choose an orthonormal basis for $T_P M$ consisting of principal directions. Having done so, we can then easily determine the curvatures of normal slices in arbitrary directions, as follows.

Proposition 2.3 (Euler's Formula). *Let $\mathbf{e}_1, \mathbf{e}_2$ be unit vectors in the principal directions at P with corresponding principal curvatures k_1 and k_2. Suppose $\mathbf{V} = \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2$ for some $\theta \in [0, 2\pi)$, as pictured in Figure 2.3. Then $\mathrm{II}_P(\mathbf{V},\mathbf{V}) = k_1\cos^2\theta + k_2\sin^2\theta$.*

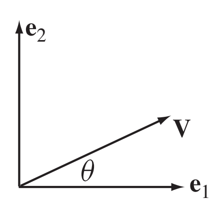

FIGURE 2.3

Proof. This is a straightforward computation: Since $S_P(\mathbf{e}_i) = k_i\mathbf{e}_i$ for $i = 1, 2$, we have

$$\mathrm{II}_P(\mathbf{V},\mathbf{V}) = S_P(\mathbf{V})\cdot\mathbf{V} = S_P(\cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2)\cdot(\cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2)$$

$$= (\cos\theta k_1\mathbf{e}_1 + \sin\theta k_2\mathbf{e}_2)\cdot(\cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2) = k_1\cos^2\theta + k_2\sin^2\theta,$$

as required. □

On a sphere, all normal slices have the same (nonzero) curvature. On the other hand, if we look carefully at Figure 2.2, we see that certain normal slices of a saddle surface are true lines. This leads us to make the following

Definition. If the normal slice in direction $\mathbf{V}$ has zero curvature, i.e., if $\mathrm{II}_P(\mathbf{V},\mathbf{V}) = 0$, then we call $\mathbf{V}$ an *asymptotic direction*.[4] A curve in M is called an *asymptotic curve* if its tangent vector at each point is an asymptotic direction.

Example 3. If a surface M contains a line, that line is an asymptotic curve. For the normal slice in the direction of the line contains the line (and perhaps other things far away), which, of course, has zero curvature. ▽

Corollary 2.4. *There is an asymptotic direction at P if and only if $k_1k_2 \le 0$.*

Proof. $k_2 = 0$ if and only if $\mathbf{e}_2$ is an asymptotic direction. Now suppose $k_2 \neq 0$. If $\mathbf{V}$ is a unit asymptotic vector making angle θ with $\mathbf{e}_1$, then we have $k_1\cos^2\theta + k_2\sin^2\theta = 0$, and so $\tan^2\theta = -k_1/k_2$, so $k_1k_2 \le 0$. Conversely, if $k_1k_2 < 0$, take θ with $\tan\theta = \pm\sqrt{-k_1/k_2}$, and then $\mathbf{V}$ is an asymptotic direction. □

Example 4. We consider the helicoid, as pictured in Figure 1.2. It is a ruled surface and so the rulings are asymptotic curves. What is quite less obvious is that the family of helices on the surface are also asymptotic curves. But, as we see in Figure 2.4, the normal slice tangent to the helix at P has an inflection

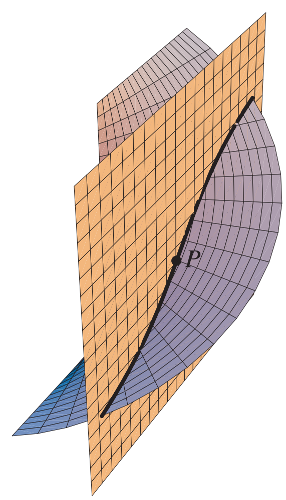

FIGURE 2.4

point at P, and therefore the helix is an asymptotic curve. We ask the reader to check this by calculation in Exercise 5. ▽

[4] Of course, $\mathbf{V} \neq \mathbf{0}$ here. See Exercise 22 for an explanation of this terminology.

It is also an immediate consequence of Proposition 2.3 that the principal curvatures are the maximum and minimum (signed) curvatures of the various normal slices. Assume $k_2 \le k_1$. Then

$$k_1 \cos^2\theta + k_2 \sin^2\theta = k_1(1-\sin^2\theta) + k_2\sin^2\theta = k_1 + (k_2-k_1)\sin^2\theta \le k_1$$

(and, similarly, $\ge k_2$). Moreover, as the Spectral Theorem tells us, the maximum and minimum occur at right angles to one another. Looking back at Figure 2.2, where the slices are taken at angles in increments of $\pi/8$, we see that the normal slices that are "most curved" appear in the third and seventh frames; the asymptotic directions appear in the second and fourth frames. (Cf. Exercise 8.)

Next we come to one of the most important concepts in the geometry of surfaces:

Definition. The product of the principal curvatures is called the *Gaussian curvature*: $K = \det S_P = k_1k_2$. The average of the principal curvatures is called the *mean curvature*: $H = \frac{1}{2}\operatorname{tr} S_P = \frac{1}{2}(k_1+k_2)$. We say M is a *minimal surface* if $H = 0$ and *flat* if $K = 0$.

Note that whereas the signs of the principal curvatures change if we reverse the direction of the unit normal $\mathbf{n}$, the Gaussian curvature K, being the product of both, is independent of the choice of unit normal. (And the sign of the mean curvature depends on the choice.)

Example 5. It follows from our comments in Example 1 that both a plane and a cylinder are flat surfaces: In the former case, $S_P = \mathrm{O}$ for all P, and, in the latter, $\det S_P = 0$ for all P since the shape operator is singular. ∇

Example 6. Consider the saddle surface $\mathbf{x}(u,v) = (u,v,uv)$. We compute:

$$\begin{aligned} \mathbf{x}_u &= (1,0,v) & \mathbf{x}_{uu} &= (0,0,0)\\ \mathbf{x}_v &= (0,1,u) & \mathbf{x}_{uv} &= (0,0,1)\\ \mathbf{n} &= \frac{1}{\sqrt{1+u^2+v^2}}(-v,-u,1) & \mathbf{x}_{vv} &= (0,0,0), \end{aligned}$$

and so

$$E = 1+v^2, \quad F = uv, \quad G = 1+u^2, \quad \text{and} \quad \ell = n = 0, m = \frac{1}{\sqrt{1+u^2+v^2}}.$$

Thus, with $P = \mathbf{x}(u,v)$, we have

$$\mathrm{I}_P = \begin{bmatrix} 1+v^2 & uv \\ uv & 1+u^2 \end{bmatrix} \quad \text{and} \quad \mathrm{II}_P = \frac{1}{\sqrt{1+u^2+v^2}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

so the matrix of the shape operator with respect to the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ is given by

$$S_P = \mathrm{I}_P^{-1}\mathrm{II}_P = \frac{1}{(1+u^2+v^2)^{3/2}}\begin{bmatrix} -uv & 1+u^2 \\ 1+v^2 & -uv \end{bmatrix}.$$

(Note that this matrix is, in general, not symmetric.)

With a bit of calculation, we determine that the principal curvatures (eigenvalues) are

$$k_1 = \frac{-uv + \sqrt{(1+u^2)(1+v^2)}}{(1+u^2+v^2)^{3/2}} \quad \text{and} \quad k_2 = \frac{-uv - \sqrt{(1+u^2)(1+v^2)}}{(1+u^2+v^2)^{3/2}},$$