It is also an immediate consequence of Proposition 2.3 that the principal curvatures are the maximum and minimum (signed) curvatures of the various normal slices. Assume $k_2 \le k_1$. Then

$$k_1 \cos^2\theta + k_2 \sin^2\theta = k_1(1-\sin^2\theta) + k_2\sin^2\theta = k_1 + (k_2 - k_1)\sin^2\theta \le k_1$$

(and, similarly, $\ge k_2$). Moreover, as the Spectral Theorem tells us, the maximum and minimum occur at right angles to one another. Looking back at Figure 2.2, where the slices are taken at angles in increments of $\pi/8$, we see that the normal slices that are "most curved" appear in the third and seventh frames; the asymptotic directions appear in the second and fourth frames. (Cf. Exercise 8.)

Next we come to one of the most important concepts in the geometry of surfaces:

Definition. The product of the principal curvatures is called the *Gaussian curvature*: $K = \det S_P = k_1k_2$. The average of the principal curvatures is called the *mean curvature*: $H = \frac{1}{2}\operatorname{tr} S_P = \frac{1}{2}(k_1 + k_2)$. We say M is a *minimal surface* if $H = 0$ and *flat* if $K = 0$.

Note that whereas the signs of the principal curvatures change if we reverse the direction of the unit normal $\mathbf{n}$, the Gaussian curvature K, being the product of both, is independent of the choice of unit normal. (And the sign of the mean curvature depends on the choice.)

Example 5. It follows from our comments in Example 1 that both a plane and a cylinder are flat surfaces: In the former case, $S_P = \mathrm{O}$ for all P, and, in the latter, $\det S_P = 0$ for all P since the shape operator is singular. ∇

Example 6. Consider the saddle surface $\mathbf{x}(u,v) = (u, v, uv)$. We compute:

$$\begin{aligned}
\mathbf{x}_u &= (1,0,v) & \mathbf{x}_{uu} &= (0,0,0)\\
\mathbf{x}_v &= (0,1,u) & \mathbf{x}_{uv} &= (0,0,1)\\
\mathbf{n} &= \frac{1}{\sqrt{1+u^2+v^2}}(-v,-u,1) & \mathbf{x}_{vv} &= (0,0,0),
\end{aligned}$$

and so

$$E = 1+v^2, \quad F = uv, \quad G = 1+u^2, \quad \text{and} \quad \ell = n = 0, m = \frac{1}{\sqrt{1+u^2+v^2}}.$$

Thus, with $P = \mathbf{x}(u,v)$, we have

$$\mathrm{I}_P = \begin{bmatrix} 1+v^2 & uv \\ uv & 1+u^2 \end{bmatrix} \quad \text{and} \quad \mathrm{II}_P = \frac{1}{\sqrt{1+u^2+v^2}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

so the matrix of the shape operator with respect to the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ is given by

$$S_P = \mathrm{I}_P^{-1}\mathrm{II}_P = \frac{1}{(1+u^2+v^2)^{3/2}}\begin{bmatrix} -uv & 1+u^2 \\ 1+v^2 & -uv \end{bmatrix}.$$

(Note that this matrix is, in general, not symmetric.)

With a bit of calculation, we determine that the principal curvatures (eigenvalues) are

$$k_1 = \frac{-uv + \sqrt{(1+u^2)(1+v^2)}}{(1+u^2+v^2)^{3/2}} \quad \text{and} \quad k_2 = \frac{-uv - \sqrt{(1+u^2)(1+v^2)}}{(1+u^2+v^2)^{3/2}},$$

and $K = \det S_P = -1/(1+u^2+v^2)^2$. Note from the form of II_P that the u- and v-curves are asymptotic curves, as should be evident from the fact that these are lines. With a bit more work, we determine that the principal directions, i.e., the eigenvectors of S_P, are the vectors

$$\sqrt{1+u^2}\,\mathbf{x}_u \pm \sqrt{1+v^2}\,\mathbf{x}_v .$$

(It is worth checking that these vectors are, in fact, orthogonal.) The corresponding curves in the uv-plane have tangent vectors $\left(\sqrt{1+u^2}, \pm\sqrt{1+v^2}\right)$ and must therefore be solutions of the differential equation

$$\frac{dv}{du} = \pm\frac{\sqrt{1+v^2}}{\sqrt{1+u^2}}.$$

If we substitute $v = \sinh q$, $\int dv/\sqrt{1+v^2} = \int dq = q = \operatorname{arcsinh} v$, so, separating variables, we obtain

$$\int \frac{dv}{\sqrt{1+v^2}} = \pm\int \frac{du}{\sqrt{1+u^2}}; \quad \text{i.e.,} \quad \operatorname{arcsinh} v = \pm \operatorname{arcsinh} u + c.$$

Since $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$, we obtain

$$v = \sinh(\pm \operatorname{arcsinh} u + c) = \pm(\cosh c)u + (\sinh c)\sqrt{1+u^2}.$$

When $c = 0$, we get $v = \pm u$ (as should be expected on geometric grounds). As c varies through nonzero values, we obtain a family of hyperbolas. Some typical lines of curvature on the saddle surface are indicated in Figure 2.5. ∇

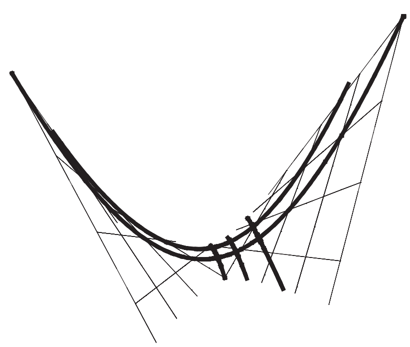

FIGURE 2.5

Definition. Fix $P \in M$. We say P is an *umbilic*[5] if $k_1 = k_2$. If $k_1 = k_2 = 0$, we say P is a *planar point*. If $K = 0$ but P is not a planar point, we say P is a *parabolic point*. If $K > 0$, we say P is an *elliptic point*, and if $K < 0$, we say P is a *hyperbolic point*.

Example 7. On the "outside" of a torus (see Figure 1.3), all the normal slices curve in the same direction, so these are elliptic points. Now imagine laying a plane on top of a torus; it is tangent to the torus along the "top circle," and so the unit normal to the surface stays constant as we move around this curve. For any point P on this circle and $\mathbf{V}$ tangent to the circle, we have $S_P(\mathbf{V}) = -D_{\mathbf{V}}\mathbf{n} = \mathbf{0}$, so $\mathbf{V}$ is a principal direction with corresponding principal curvature 0. Thus, these are parabolic points. On the other hand, consider a point P on the innermost band of the torus. At such a point the surface looks saddle-like; that is, with the unit normal as pictured in Figure 2.6, the horizontal circle (going around the inside of the torus) is a

[5]From the Latin *umbilīcus*, navel.