and $K = \det S_P = -1/(1+u^2+v^2)^2$. Note from the form of II_P that the u- and v-curves are asymptotic curves, as should be evident from the fact that these are lines. With a bit more work, we determine that the principal directions, i.e., the eigenvectors of S_P, are the vectors

$$\sqrt{1+u^2}\mathbf{x}_u \pm \sqrt{1+v^2}\mathbf{x}_v.$$

(It is worth checking that these vectors are, in fact, orthogonal.) The corresponding curves in the uv-plane have tangent vectors $\big(\sqrt{1+u^2}, \pm\sqrt{1+v^2}\big)$ and must therefore be solutions of the differential equation

$$\frac{dv}{du} = \pm\frac{\sqrt{1+v^2}}{\sqrt{1+u^2}}.$$

If we substitute $v = \sinh q$, $\int dv/\sqrt{1+v^2} = \int dq = q = \operatorname{arcsinh} v$, so, separating variables, we obtain

$$\int \frac{dv}{\sqrt{1+v^2}} = \pm\int \frac{du}{\sqrt{1+u^2}}; \quad \text{i.e.,} \quad \operatorname{arcsinh} v = \pm \operatorname{arcsinh} u + c.$$

Since $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$, we obtain

$$v = \sinh(\pm \operatorname{arcsinh} u + c) = \pm(\cosh c)u + (\sinh c)\sqrt{1+u^2}.$$

When $c = 0$, we get $v = \pm u$ (as should be expected on geometric grounds). As c varies through nonzero values, we obtain a family of hyperbolas. Some typical lines of curvature on the saddle surface are indicated in Figure 2.5. ∇

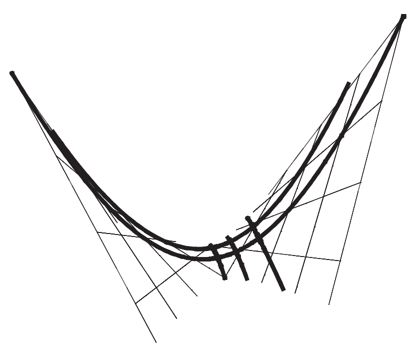

FIGURE 2.5

Definition. Fix $P \in M$. We say P is an *umbilic*[5] if $k_1 = k_2$. If $k_1 = k_2 = 0$, we say P is a *planar point*. If $K = 0$ but P is not a planar point, we say P is a *parabolic point*. If $K > 0$, we say P is an *elliptic point*, and if $K < 0$, we say P is a *hyperbolic point*.

Example 7. On the "outside" of a torus (see Figure 1.3), all the normal slices curve in the same direction, so these are elliptic points. Now imagine laying a plane on top of a torus; it is tangent to the torus along the "top circle," and so the unit normal to the surface stays constant as we move around this curve. For any point P on this circle and $\mathbf{V}$ tangent to the circle, we have $S_P(\mathbf{V}) = -D_{\mathbf{V}}\mathbf{n} = \mathbf{0}$, so $\mathbf{V}$ is a principal direction with corresponding principal curvature 0. Thus, these are parabolic points. On the other hand, consider a point P on the innermost band of the torus. At such a point the surface looks saddle-like; that is, with the unit normal as pictured in Figure 2.6, the horizontal circle (going around the inside of the torus) is a

[5]From the Latin *umbilīcus*, navel.

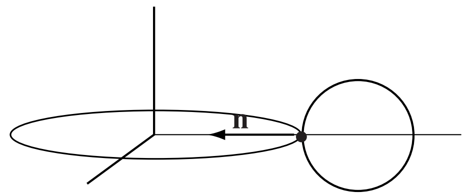

FIGURE 2.6

line of curvature with positive principal curvature, and the vertical circle is a line of curvature with negative principal curvature. Thus, the points on the inside of the torus are hyperbolic points. ∇

Remark. Gauss's original interpretation of Gaussian curvature was the following: Imagine a small curvilinear rectangle $\mathcal{P}$ at $P \in M$ with sides h_1 and h_2 along principal directions. Then, since the principal directions are eigenvectors of the shape operator, the image of $\mathcal{P}$ under the Gauss map is nearly a small curvilinear rectangle at $\mathbf{n}(P) \in \Sigma$ with sides k_1h_1 and k_2h_2. Thus, $K = k_1k_2$ is the factor by which $\mathbf{n}$ distorts signed area as it maps M to Σ. (Note that for a cylinder, the rectangle collapses to a line segment; for a saddle surface, orientation is reversed by $\mathbf{n}$ and so the Gaussian curvature is negative.)

Let's close this section by revisiting our discussion of the curvature of normal slices. Suppose $\boldsymbol{\alpha}$ is an arclength-parametrized curve lying on M with $\boldsymbol{\alpha}(0) = P$ and $\boldsymbol{\alpha}'(0) = \mathbf{V}$. Then the calculation in formula (†) on p. 45 shows that

$$\mathrm{II}_P(\mathbf{V}, \mathbf{V}) = \kappa\mathbf{N} \cdot \mathbf{n};$$

i.e., $\mathrm{II}_P(\mathbf{V}, \mathbf{V})$ gives the component of the curvature vector $\kappa\mathbf{N}$ of $\boldsymbol{\alpha}$ *normal* to the surface M at P, which we denote by κ_n and call the *normal curvature* of $\boldsymbol{\alpha}$ at P. What is remarkable about this formula is that it shows that the normal curvature depends only on the direction of $\boldsymbol{\alpha}$ at P and otherwise not on the curve. (For the case of the normal slice, the normal curvature is, up to a sign, all the curvature.) What's more, κ_n can be computed just from the second fundamental form II of M. We immediately deduce the following

Proposition 2.5 (Meusnier's Formula). *Let $\boldsymbol{\alpha}$ be a curve on M passing through P with unit tangent vector $\mathbf{V}$. Then*

$$\mathrm{II}_P(\mathbf{V}, \mathbf{V}) = \kappa_n = \kappa \cos\phi,$$

where ϕ is the angle between the principal normal, $\mathbf{N}$, of $\boldsymbol{\alpha}$ and the surface normal, $\mathbf{n}$, at P.

In particular, if $\boldsymbol{\alpha}$ is an asymptotic curve, then its normal curvature is 0 at each point. This means that, so long as $\kappa \neq 0$, its principal normal is always orthogonal to the surface normal, i.e., always tangent to the surface.

Example 8. Let's now investigate a very interesting surface, called the *pseudosphere*, as shown in Figure 2.7. It is the surface of revolution obtained by rotating the tractrix (see Example 2 of Chapter 1, Section 1) about the x-axis, and so it is parametrized by

$$\mathbf{x}(u, v) = (u - \tanh u, \operatorname{sech} u \cos v, \operatorname{sech} u \sin v), \qquad u > 0,\ v \in [0, 2\pi).$$

Note that the circles (of revolution) are lines of curvature: Either apply Exercise 15 or observe, directly, that the only component of the surface normal that changes as we move around the circle is normal to the circle

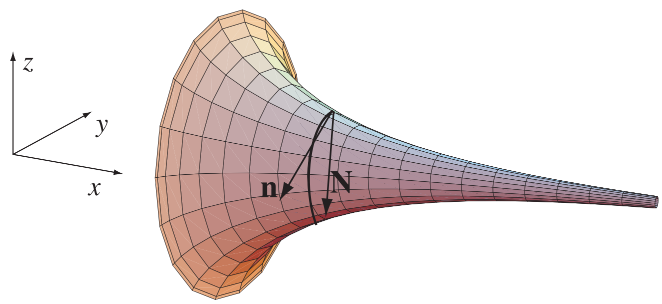

FIGURE 2.7

in the plane of the circle. Similarly, the various tractrices are lines of curvature: In the plane of one tractrix, the surface normal and the curve normal agree.

Now, by Exercise 1.2.5, the curvature of the tractrix is $\kappa = 1/\sinh u$; since $\mathbf{N} = -\mathbf{n}$ along this curve, we have $k_1 = \kappa_n = -1/\sinh u$. Now what about the circles? Here we have $\kappa = 1/\operatorname{sech} u = \cosh u$, but this is not the normal curvature. The angle ϕ between $\mathbf{N}$ and $\mathbf{n}$ is the supplement of the angle θ we see in Figure 1.9 of Chapter 1 (to see why, see Figure 2.8). Thus, by Meusnier's Formula, Proposition 2.5,

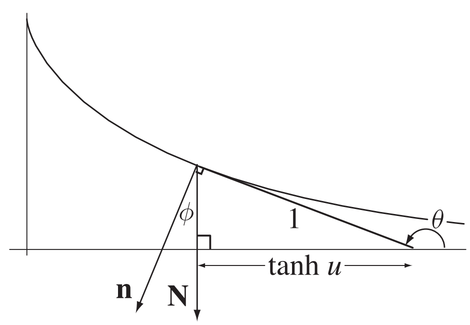

FIGURE 2.8

we have $k_2 = \kappa_n = \kappa \cos\phi = (\cosh u)(\tanh u) = \sinh u$. Amazingly, then, we have $K = k_1 k_2 = (-1/\sinh u)(\sinh u) = -1$. ∇

Example 9. Let's now consider the case of a general surface of revolution, parametrized as in Example 2 of Section 1, by

$$\mathbf{x}(u,v) = \big(f(u)\cos v, f(u)\sin v, g(u)\big),$$

where $f'(u)^2 + g'(u)^2 = 1$. Recall that the u-curves are called *meridians* and the v-curves are called *parallels*. Then

$$\begin{aligned}
\mathbf{x}_u &= \big(f'(u)\cos v, f'(u)\sin v, g'(u)\big)\\
\mathbf{x}_v &= \big(-f(u)\sin v, f(u)\cos v, 0\big)\\
\mathbf{n} &= \big(-g'(u)\cos v, -g'(u)\sin v, f'(u)\big)\\
\mathbf{x}_{uu} &= \big(f''(u)\cos v, f''(u)\sin v, g''(u)\big)\\
\mathbf{x}_{uv} &= \big(-f'(u)\sin v, f'(u)\cos v, 0\big)\\
\mathbf{x}_{vv} &= \big(-f(u)\cos v, -f(u)\sin v, 0\big),
\end{aligned}$$

and so we have

$$E = 1, \quad F = 0, \quad G = f(u)^2, \quad \text{and} \quad \ell = f'(u)g''(u) - f''(u)g'(u), \quad m = 0, \quad n = f(u)g'(u).$$

By Exercise 2.2.1, then $k_1 = f'(u)g''(u) - f''(u)g'(u)$ and $k_2 = g'(u)/f(u)$. Thus,

$$K = k_1k_2 = \big(f'(u)g''(u) - f''(u)g'(u)\big)\frac{g'(u)}{f(u)} = -\frac{f''(u)}{f(u)},$$

since from $f'(u)^2 + g'(u)^2 = 1$ we deduce that $f'(u)f''(u) + g'(u)g''(u) = 0$, and so

$$f'(u)g'(u)g''(u) - f''(u)g'(u)^2 = -(f'(u)^2 + g'(u)^2)f''(u) = -f''(u).$$

Note, as we observed in the special case of Example 8, that on every surface of revolution, the meridians and the parallels are lines of curvature. ∇

EXERCISES 2.2

*1. Check that if there are no umbilic points and the parameter curves are lines of curvature, then $F = m = 0$ and we have the principal curvatures $k_1 = \ell/E$ and $k_2 = n/G$. Conversely, prove that if $F = m = 0$, then the parameter curves are lines of curvature.

♯2. a. Show that the matrix representing the linear map $S_P\colon T_PM \to T_PM$ with respect to the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ is

$$\mathrm{I}_P^{-1}\mathrm{II}_P = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} \ell & m \\ m & n \end{bmatrix}.$$

(Hint: Write $S_P(\mathbf{x}_u) = a\mathbf{x}_u + b\mathbf{x}_v$ and $S_P(\mathbf{x}_v) = c\mathbf{x}_u + d\mathbf{x}_v$, and use the definition of ℓ, m, and n to get a system of linear equations for a, b, c, and d.)

b. Deduce that $K = \dfrac{\ell n - m^2}{EG - F^2}$.

3. Compute the second fundamental form II_P of the following parametrized surfaces. Then calculate the matrix of the shape operator, and determine H and K.
   a. the cylinder: $\mathbf{x}(u, v) = (a\cos u, a\sin u, v)$
   *b. the torus: $\mathbf{x}(u, v) = ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u)$ $(0 < b < a)$
   c. the helicoid: $\mathbf{x}(u, v) = (u\cos v, u\sin v, bv)$
   *d. the catenoid: $\mathbf{x}(u, v) = a(\cosh u\cos v, \cosh u\sin v, u)$
   e. the Mercator parametrization of the sphere: $\mathbf{x}(u, v) = (\operatorname{sech} u\cos v, \operatorname{sech} u\sin v, \tanh u)$
   f. Enneper's surface: $\mathbf{x}(u, v) = (u - u^3/3 + uv^2, v - v^3/3 + u^2v, u^2 - v^2)$

4. Find the principal curvatures, the principal directions, and asymptotic directions (when they exist) for each of the surfaces in Exercise 3. Identify the lines of curvature and asymptotic curves when possible.

*5. Prove by calculation that any one of the helices $\boldsymbol{\alpha}(t) = (a\cos t, a\sin t, bt)$ is an asymptotic curve on the helicoid given in Example 1(b) of Section 1. Also, calculate how the surface normal $\mathbf{n}$ changes as one moves along a ruling, and use this to explain why the rulings are asymptotic curves as well.