$\mathcal{F}(0) = \mathcal{F}'(0) = \cdots = \mathcal{F}^{(k-1)}(0) = 0$. (Such a line is to be visualized as the limit of lines that intersect M at P and at $k-1$ other points that approach P.)

a. Show that L has 2-point contact with M at P if and only if L is tangent to M at P, i.e., $L \subset T_P M$.

b. Show that L has 3-point contact with M at P if and only if L is an asymptotic direction at P. (Hint: It may be helpful to follow the setup of Exercise 21.)

c. (Challenge) Assume P is a hyperbolic point. What does it mean for L to have 4-point contact with M at P?

3. The Codazzi and Gauss Equations and the Fundamental Theorem of Surface Theory

We now wish to proceed towards a deeper understanding of Gaussian curvature. We have to this point considered only the normal components of the second derivatives $\mathbf{x}_{uu}$, $\mathbf{x}_{uv}$, and $\mathbf{x}_{vv}$. Now let's consider them *in toto*. Since $\{\mathbf{x}_u, \mathbf{x}_v, \mathbf{n}\}$ gives a basis for $\mathbb{R}^3$, there are functions Γ^u_{uu}, Γ^v_{uu}, $\Gamma^u_{uv} = \Gamma^u_{vu}$, $\Gamma^v_{uv} = \Gamma^v_{vu}$, Γ^u_{vv}, and Γ^v_{vv} so that

$$(\dagger) \qquad \begin{aligned} \mathbf{x}_{uu} &= \Gamma^u_{uu}\mathbf{x}_u + \Gamma^v_{uu}\mathbf{x}_v + \ell\mathbf{n} \\ \mathbf{x}_{uv} &= \Gamma^u_{uv}\mathbf{x}_u + \Gamma^v_{uv}\mathbf{x}_v + m\mathbf{n} \\ \mathbf{x}_{vv} &= \Gamma^u_{vv}\mathbf{x}_u + \Gamma^v_{vv}\mathbf{x}_v + n\mathbf{n}. \end{aligned}$$

(Note that $\mathbf{x}_{uv} = \mathbf{x}_{vu}$ dictates the symmetries $\Gamma^{\bullet}_{uv} = \Gamma^{\bullet}_{vu}$.) The functions $\Gamma^{\bullet}_{\bullet\bullet}$ are called *Christoffel symbols*.

Example 1. Let's compute the Christoffel symbols for the usual parametrization of the sphere (see Example 1(d) on p. 37). By straightforward calculation we obtain

$$\begin{aligned} \mathbf{x}_u &= (\cos u \cos v, \cos u \sin v, -\sin u) \\ \mathbf{x}_v &= (-\sin u \sin v, \sin u \cos v, 0) \\ \mathbf{x}_{uu} &= (-\sin u \cos v, -\sin u \sin v, -\cos u) = -\mathbf{x}(u, v) \\ \mathbf{x}_{uv} &= (-\cos u \sin v, \cos u \cos v, 0) \\ \mathbf{x}_{vv} &= (-\sin u \cos v, -\sin u \sin v, 0) = -\sin u(\cos v, \sin v, 0). \end{aligned}$$

(Note that the u-curves are great circles, parametrized by arclength, so it is no surprise that the acceleration vector $\mathbf{x}_{uu}$ is inward-pointing of length 1. The v-curves are latitude circles of radius $\sin u$, so, similarly, the acceleration vector $\mathbf{x}_{vv}$ points inwards towards the center of the respective circle.)

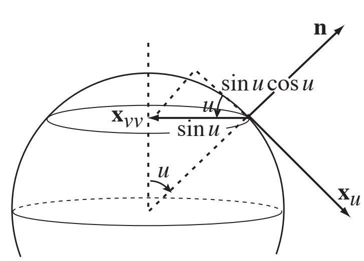

FIGURE 3.1

Since $\mathbf{x}_{uu}$ lies entirely in the direction of $\mathbf{n}$, we have $\Gamma^u_{uu} = \Gamma^v_{uu} = 0$. Now, by inspection, $\mathbf{x}_{uv} = \cot u \mathbf{x}_v$, so $\Gamma^u_{uv} = 0$ and $\Gamma^v_{uv} = \cot u$. Last, as we can see in Figure 3.1, we have $\mathbf{x}_{vv} = -\sin u \cos u \mathbf{x}_u - \sin^2 u \mathbf{n}$, so $\Gamma^u_{vv} = -\sin u \cos u$ and $\Gamma^v_{vv} = 0$. ▽

Now, dotting the equations in (†) with $\mathbf{x}_u$ and $\mathbf{x}_v$ gives

$$\begin{aligned}
\mathbf{x}_{uu} \cdot \mathbf{x}_u &= \Gamma^u_{uu} E + \Gamma^v_{uu} F \\
\mathbf{x}_{uu} \cdot \mathbf{x}_v &= \Gamma^u_{uu} F + \Gamma^v_{uu} G \\[1ex]
\mathbf{x}_{uv} \cdot \mathbf{x}_u &= \Gamma^u_{uv} E + \Gamma^v_{uv} F \\
\mathbf{x}_{uv} \cdot \mathbf{x}_v &= \Gamma^u_{uv} F + \Gamma^v_{uv} G \\[1ex]
\mathbf{x}_{vv} \cdot \mathbf{x}_u &= \Gamma^u_{vv} E + \Gamma^v_{vv} F \\
\mathbf{x}_{vv} \cdot \mathbf{x}_v &= \Gamma^u_{vv} F + \Gamma^v_{vv} G.
\end{aligned}$$

Now observe that

$$(\spadesuit) \qquad \begin{aligned}
\mathbf{x}_{uu} \cdot \mathbf{x}_u &= \tfrac{1}{2}(\mathbf{x}_u \cdot \mathbf{x}_u)_u = \tfrac{1}{2} E_u \\
\mathbf{x}_{uv} \cdot \mathbf{x}_u &= \tfrac{1}{2}(\mathbf{x}_u \cdot \mathbf{x}_u)_v = \tfrac{1}{2} E_v \\
\mathbf{x}_{uv} \cdot \mathbf{x}_v &= \tfrac{1}{2}(\mathbf{x}_v \cdot \mathbf{x}_v)_u = \tfrac{1}{2} G_u \\
\mathbf{x}_{uu} \cdot \mathbf{x}_v &= (\mathbf{x}_u \cdot \mathbf{x}_v)_u - \mathbf{x}_u \cdot \mathbf{x}_{uv} = F_u - \tfrac{1}{2} E_v \\
\mathbf{x}_{vv} \cdot \mathbf{x}_u &= (\mathbf{x}_u \cdot \mathbf{x}_v)_v - \mathbf{x}_{uv} \cdot \mathbf{x}_v = F_v - \tfrac{1}{2} G_u \\
\mathbf{x}_{vv} \cdot \mathbf{x}_v &= \tfrac{1}{2}(\mathbf{x}_v \cdot \mathbf{x}_v)_v = \tfrac{1}{2} G_v
\end{aligned}$$

Thus, we can rewrite our equations as follows:

$$(\ddagger) \qquad \begin{aligned}
\begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} \Gamma^u_{uu} \\ \Gamma^v_{uu} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} E_u \\ F_u - \frac{1}{2} E_v \end{bmatrix} \quad &\Longrightarrow \quad \begin{bmatrix} \Gamma^u_{uu} \\ \Gamma^v_{uu} \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{2} E_u \\ F_u - \frac{1}{2} E_v \end{bmatrix} \\
\begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} \Gamma^u_{uv} \\ \Gamma^v_{uv} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} E_v \\ \frac{1}{2} G_u \end{bmatrix} \quad &\Longrightarrow \quad \begin{bmatrix} \Gamma^u_{uv} \\ \Gamma^v_{uv} \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{2} E_v \\ \frac{1}{2} G_u \end{bmatrix} \\
\begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} \Gamma^u_{vv} \\ \Gamma^v_{vv} \end{bmatrix} = \begin{bmatrix} F_v - \frac{1}{2} G_u \\ \frac{1}{2} G_v \end{bmatrix} \quad &\Longrightarrow \quad \begin{bmatrix} \Gamma^u_{vv} \\ \Gamma^v_{vv} \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} F_v - \frac{1}{2} G_u \\ \frac{1}{2} G_v \end{bmatrix}.
\end{aligned}$$

What is quite remarkable about these formulas is that the Christoffel symbols, which tell us about the tangential component of the second derivatives $\mathbf{x}_{\bullet\bullet}$, can be computed *just* from knowing E, F, and G, i.e., the first fundamental form.

Example 2. Let's now recompute the Christoffel symbols of the unit sphere and compare our answers with Example 1. Since $E = 1$, $F = 0$, and $G = \sin^2 u$, we have

$$\begin{aligned}
\begin{bmatrix} \Gamma^u_{uu} \\ \Gamma^v_{uu} \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & \csc^2 u \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} \Gamma^u_{uv} \\ \Gamma^v_{uv} \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & \csc^2 u \end{bmatrix} \begin{bmatrix} 0 \\ \sin u \cos u \end{bmatrix} = \begin{bmatrix} 0 \\ \cot u \end{bmatrix}
\end{aligned}$$

$$\begin{bmatrix} \Gamma^u_{vv} \\ \Gamma^v_{vv} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \csc^2 u \end{bmatrix} \begin{bmatrix} -\sin u \cos u \\ 0 \end{bmatrix} = \begin{bmatrix} -\sin u \cos u \\ 0 \end{bmatrix}.$$

Thus, the only nonzero Christoffel symbols are $\Gamma^v_{uv} = \Gamma^v_{vu} = \cot u$ and $\Gamma^u_{vv} = -\sin u \cos u$, as before. ∇

By Exercise 2.2.2, the matrix of the shape operator S_P with respect to the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ is

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} \ell & m \\ m & n \end{bmatrix} = \frac{1}{EG - F^2} \begin{bmatrix} \ell G - mF & mG - nF \\ -\ell F + mE & -mF + nE \end{bmatrix}.$$

Note that these coefficients tell us the derivatives of $\mathbf{n}$ with respect to u and v:

$$(\dagger\dagger) \qquad \begin{aligned} \mathbf{n}_u &= D_{\mathbf{x}_u}\mathbf{n} = -S_P(\mathbf{x}_u) = -(a\mathbf{x}_u + b\mathbf{x}_v) \\ \mathbf{n}_v &= D_{\mathbf{x}_v}\mathbf{n} = -S_P(\mathbf{x}_v) = -(c\mathbf{x}_u + d\mathbf{x}_v). \end{aligned}$$

We now differentiate the equations (†) again and use equality of mixed partial derivatives. To start, we have

$$\begin{aligned} \mathbf{x}_{uuv} &= (\Gamma^u_{uu})_v\mathbf{x}_u + \Gamma^u_{uu}\mathbf{x}_{uv} + (\Gamma^v_{uu})_v\mathbf{x}_v + \Gamma^v_{uu}\mathbf{x}_{vv} + \ell_v\mathbf{n} + \ell\mathbf{n}_v \\ &= (\Gamma^u_{uu})_v\mathbf{x}_u + \Gamma^u_{uu}\big(\Gamma^u_{uv}\mathbf{x}_u + \Gamma^v_{uv}\mathbf{x}_v + m\mathbf{n}\big) + (\Gamma^v_{uu})_v\mathbf{x}_v + \Gamma^v_{uu}\big(\Gamma^u_{vv}\mathbf{x}_u + \Gamma^v_{vv}\mathbf{x}_v + n\mathbf{n}\big) \\ &\qquad + \ell_v\mathbf{n} - \ell(c\mathbf{x}_u + d\mathbf{x}_v) \\ &= \big((\Gamma^u_{uu})_v + \Gamma^u_{uu}\Gamma^u_{uv} + \Gamma^v_{uu}\Gamma^u_{vv} - \ell c\big)\mathbf{x}_u + \big((\Gamma^v_{uu})_v + \Gamma^u_{uu}\Gamma^v_{uv} + \Gamma^v_{uu}\Gamma^v_{vv} - \ell d\big)\mathbf{x}_v \\ &\qquad + \big(\Gamma^u_{uu}m + \Gamma^v_{uu}n + \ell_v\big)\mathbf{n}, \end{aligned}$$

and, similarly,

$$\begin{aligned} \mathbf{x}_{uvu} &= \big((\Gamma^u_{uv})_u + \Gamma^u_{uv}\Gamma^u_{uu} + \Gamma^v_{uv}\Gamma^u_{uv} - ma\big)\mathbf{x}_u + \big((\Gamma^v_{uv})_u + \Gamma^u_{uv}\Gamma^v_{uu} + \Gamma^v_{uv}\Gamma^v_{uv} - mb\big)\mathbf{x}_v \\ &\qquad + \big(\ell\Gamma^u_{uv} + m\Gamma^v_{uv} + m_u\big)\mathbf{n}. \end{aligned}$$

Since $\mathbf{x}_{uuv} = \mathbf{x}_{uvu}$, we compare the indicated components and obtain:

$$(\diamondsuit) \qquad \begin{aligned} &(\mathbf{x}_u): && (\Gamma^u_{uu})_v + \Gamma^v_{uu}\Gamma^u_{vv} - \ell c = (\Gamma^u_{uv})_u + \Gamma^v_{uv}\Gamma^u_{uv} - ma \\ &(\mathbf{x}_v): && (\Gamma^v_{uu})_v + \Gamma^u_{uu}\Gamma^v_{uv} + \Gamma^v_{uu}\Gamma^v_{vv} - \ell d = (\Gamma^v_{uv})_u + \Gamma^u_{uv}\Gamma^v_{uu} + \Gamma^v_{uv}\Gamma^v_{uv} - mb \\ &(\mathbf{n}): && \ell_v + m\Gamma^u_{uu} + n\Gamma^v_{uu} = m_u + \ell\Gamma^u_{uv} + m\Gamma^v_{uv}. \end{aligned}$$

Analogously, comparing the indicated components of $\mathbf{x}_{uvv} = \mathbf{x}_{vvu}$, we find:

$$\begin{aligned} &(\mathbf{x}_u): && (\Gamma^u_{uv})_v + \Gamma^u_{uv}\Gamma^u_{uv} + \Gamma^v_{uv}\Gamma^u_{vv} - mc = (\Gamma^u_{vv})_u + \Gamma^u_{vv}\Gamma^u_{uu} + \Gamma^v_{vv}\Gamma^u_{uv} - na \\ &(\mathbf{x}_v): && (\Gamma^v_{uv})_v + \Gamma^u_{uv}\Gamma^v_{uv} - md = (\Gamma^v_{vv})_u + \Gamma^u_{vv}\Gamma^v_{uu} - nb \\ &(\mathbf{n}): && m_v + m\Gamma^u_{uv} + n\Gamma^v_{uv} = n_u + \ell\Gamma^u_{vv} + m\Gamma^v_{vv}. \end{aligned}$$

The two equations coming from the normal component give us the

Codazzi equations

$$\begin{aligned} \ell_v - m_u &= \ell\Gamma^u_{uv} + m\big(\Gamma^v_{uv} - \Gamma^u_{uu}\big) - n\Gamma^v_{uu} \\ m_v - n_u &= \ell\Gamma^u_{vv} + m\big(\Gamma^v_{vv} - \Gamma^u_{uv}\big) - n\Gamma^v_{uv}. \end{aligned}$$