$$\begin{bmatrix}\Gamma_{vv}^{u}\\ \Gamma_{vv}^{v}\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & \csc^2 u\end{bmatrix}\begin{bmatrix}-\sin u\cos u\\ 0\end{bmatrix}=\begin{bmatrix}-\sin u\cos u\\ 0\end{bmatrix}.$$

Thus, the only nonzero Christoffel symbols are $\Gamma_{uv}^{v}=\Gamma_{vu}^{v}=\cot u$ and $\Gamma_{vv}^{u}=-\sin u\cos u$, as before. $\nabla$

By Exercise 2.2.2, the matrix of the shape operator $S_P$ with respect to the basis $\{\mathbf{x}_u,\mathbf{x}_v\}$ is

$$\begin{bmatrix}a & c\\ b & d\end{bmatrix}=\begin{bmatrix}E & F\\ F & G\end{bmatrix}^{-1}\begin{bmatrix}\ell & m\\ m & n\end{bmatrix}=\frac{1}{EG-F^2}\begin{bmatrix}\ell G-mF & mG-nF\\ -\ell F+mE & -mF+nE\end{bmatrix}.$$

Note that these coefficients tell us the derivatives of $\mathbf{n}$ with respect to $u$ and $v$:

$$(\dagger\dagger)\qquad \begin{aligned}\mathbf{n}_u &= D_{\mathbf{x}_u}\mathbf{n}=-S_P(\mathbf{x}_u)=-(a\mathbf{x}_u+b\mathbf{x}_v)\\ \mathbf{n}_v &= D_{\mathbf{x}_v}\mathbf{n}=-S_P(\mathbf{x}_v)=-(c\mathbf{x}_u+d\mathbf{x}_v).\end{aligned}$$

We now differentiate the equations (†) again and use equality of mixed partial derivatives. To start, we have

$$\begin{aligned}\mathbf{x}_{uuv} &= (\Gamma_{uu}^{u})_v\mathbf{x}_u+\Gamma_{uu}^{u}\mathbf{x}_{uv}+(\Gamma_{uu}^{v})_v\mathbf{x}_v+\Gamma_{uu}^{v}\mathbf{x}_{vv}+\ell_v\mathbf{n}+\ell\mathbf{n}_v\\ &= (\Gamma_{uu}^{u})_v\mathbf{x}_u+\Gamma_{uu}^{u}\big(\Gamma_{uv}^{u}\mathbf{x}_u+\Gamma_{uv}^{v}\mathbf{x}_v+m\mathbf{n}\big)+(\Gamma_{uu}^{v})_v\mathbf{x}_v+\Gamma_{uu}^{v}\big(\Gamma_{vv}^{u}\mathbf{x}_u+\Gamma_{vv}^{v}\mathbf{x}_v+n\mathbf{n}\big)\\ &\qquad +\ell_v\mathbf{n}-\ell(c\mathbf{x}_u+d\mathbf{x}_v)\\ &= \big((\Gamma_{uu}^{u})_v+\Gamma_{uu}^{u}\Gamma_{uv}^{u}+\Gamma_{uu}^{v}\Gamma_{vv}^{u}-\ell c\big)\mathbf{x}_u+\big((\Gamma_{uu}^{v})_v+\Gamma_{uu}^{u}\Gamma_{uv}^{v}+\Gamma_{uu}^{v}\Gamma_{vv}^{v}-\ell d\big)\mathbf{x}_v\\ &\qquad +\big(\Gamma_{uu}^{u}m+\Gamma_{uu}^{v}n+\ell_v\big)\mathbf{n},\end{aligned}$$

and, similarly,

$$\begin{aligned}\mathbf{x}_{uvu} &= \big((\Gamma_{uv}^{u})_u+\Gamma_{uv}^{u}\Gamma_{uu}^{u}+\Gamma_{uv}^{v}\Gamma_{uv}^{u}-ma\big)\mathbf{x}_u+\big((\Gamma_{uv}^{v})_u+\Gamma_{uv}^{u}\Gamma_{uu}^{v}+\Gamma_{uv}^{v}\Gamma_{uv}^{v}-mb\big)\mathbf{x}_v\\ &\qquad +\big(\ell\Gamma_{uv}^{u}+m\Gamma_{uv}^{v}+m_u\big)\mathbf{n}.\end{aligned}$$

Since $\mathbf{x}_{uuv}=\mathbf{x}_{uvu}$, we compare the indicated components and obtain:

$$(\diamond)\qquad \begin{aligned}&(\mathbf{x}_u): && (\Gamma_{uu}^{u})_v+\Gamma_{uu}^{v}\Gamma_{vv}^{u}-\ell c=(\Gamma_{uv}^{u})_u+\Gamma_{uv}^{v}\Gamma_{uv}^{u}-ma\\ &(\mathbf{x}_v): && (\Gamma_{uu}^{v})_v+\Gamma_{uu}^{u}\Gamma_{uv}^{v}+\Gamma_{uu}^{v}\Gamma_{vv}^{v}-\ell d=(\Gamma_{uv}^{v})_u+\Gamma_{uv}^{u}\Gamma_{uu}^{v}+\Gamma_{uv}^{v}\Gamma_{uv}^{v}-mb\\ &(\mathbf{n}): && \ell_v+m\Gamma_{uu}^{u}+n\Gamma_{uu}^{v}=m_u+\ell\Gamma_{uv}^{u}+m\Gamma_{uv}^{v}.\end{aligned}$$

Analogously, comparing the indicated components of $\mathbf{x}_{uvv}=\mathbf{x}_{vvu}$, we find:

$$\begin{aligned}&(\mathbf{x}_u): && (\Gamma_{uv}^{u})_v+\Gamma_{uv}^{u}\Gamma_{uv}^{u}+\Gamma_{uv}^{v}\Gamma_{vv}^{u}-mc=(\Gamma_{vv}^{u})_u+\Gamma_{vv}^{u}\Gamma_{uu}^{u}+\Gamma_{vv}^{v}\Gamma_{uv}^{u}-na\\ &(\mathbf{x}_v): && (\Gamma_{uv}^{v})_v+\Gamma_{uv}^{u}\Gamma_{uv}^{v}-md=(\Gamma_{vv}^{v})_u+\Gamma_{vv}^{u}\Gamma_{uu}^{v}-nb\\ &(\mathbf{n}): && m_v+m\Gamma_{uv}^{u}+n\Gamma_{uv}^{v}=n_u+\ell\Gamma_{vv}^{u}+m\Gamma_{vv}^{v}.\end{aligned}$$

The two equations coming from the normal component give us the

**Codazzi equations**

$$\begin{aligned}\ell_v-m_u &= \ell\Gamma_{uv}^{u}+m\big(\Gamma_{uv}^{v}-\Gamma_{uu}^{u}\big)-n\Gamma_{uu}^{v}\\ m_v-n_u &= \ell\Gamma_{vv}^{u}+m\big(\Gamma_{vv}^{v}-\Gamma_{uv}^{u}\big)-n\Gamma_{uv}^{v}.\end{aligned}$$

Using $K = \dfrac{\ell n - m^2}{EG - F^2}$ and the formulas above for $a$, $b$, $c$, and $d$, the four equations involving the $\mathbf{x}_u$ and $\mathbf{x}_v$ components yield the

**Gauss equations**

$$\begin{aligned}
EK &= \left(\Gamma_{uu}^{v}\right)_v - \left(\Gamma_{uv}^{v}\right)_u + \Gamma_{uu}^{u}\Gamma_{uv}^{v} + \Gamma_{uu}^{v}\Gamma_{vv}^{v} - \Gamma_{uv}^{u}\Gamma_{uu}^{v} - \left(\Gamma_{uv}^{v}\right)^2 \\
FK &= \left(\Gamma_{uv}^{u}\right)_u - \left(\Gamma_{uu}^{u}\right)_v + \Gamma_{uv}^{v}\Gamma_{uv}^{u} - \Gamma_{uu}^{v}\Gamma_{vv}^{u} \\
FK &= \left(\Gamma_{uv}^{v}\right)_v - \left(\Gamma_{vv}^{v}\right)_u + \Gamma_{uv}^{u}\Gamma_{uv}^{v} - \Gamma_{vv}^{u}\Gamma_{uu}^{v} \\
GK &= \left(\Gamma_{vv}^{u}\right)_u - \left(\Gamma_{uv}^{u}\right)_v + \Gamma_{vv}^{u}\Gamma_{uu}^{u} + \Gamma_{vv}^{v}\Gamma_{uv}^{u} - \left(\Gamma_{uv}^{u}\right)^2 - \Gamma_{uv}^{v}\Gamma_{vv}^{u}.
\end{aligned}$$

For example, to derive the first, we use the equation ($\diamond$) above:

$$\begin{aligned}
\left(\Gamma_{uu}^{v}\right)_v - \left(\Gamma_{uv}^{v}\right)_u + \Gamma_{uu}^{u}\Gamma_{uv}^{v} + \Gamma_{uu}^{v}\Gamma_{vv}^{v} - \Gamma_{uv}^{u}\Gamma_{uu}^{v} - \left(\Gamma_{uv}^{v}\right)^2 &= \ell d - mb \\
&= \frac{1}{EG - F^2}\big(\ell(-mF + nE) + m(\ell F - mE)\big) = \frac{E(\ell n - m^2)}{EG - F^2} = EK.
\end{aligned}$$

In an orthogonal parametrization ($F = 0$), we leave it to the reader to check in Exercise 3 that

$$(*) \qquad K = -\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right).$$

One of the crowning results of local differential geometry is the following

**Theorem 3.1** (Gauss's Theorema Egregium)**.** *The Gaussian curvature is determined by only the first fundamental form* I. *That is, $K$ can be computed from just $E$, $F$, $G$, and their first and second partial derivatives.*

**Proof.** From any of the Gauss equations, we see that $K$ can be computed by knowing any one of $E$, $F$, and $G$, together with the Christoffel symbols and their derivatives. But the equations (‡) show that the Christoffel symbols (and hence any of their derivatives) can be calculated in terms of $E$, $F$, and $G$ and their partial derivatives. $\square$

**Corollary 3.2.** *If two surfaces are locally isometric, their Gaussian curvatures at corresponding points are equal.*

For example, the plane and cylinder are locally isometric, and hence the cylinder (as we well know) is flat. We now conclude that since the Gaussian curvature of a sphere is nonzero, a sphere cannot be locally isometric to a plane. Thus, there is no way to map the earth "faithfully" (preserving distance)—even locally—on a piece of paper. In some sense, the Mercator projection (see Exercise 2.1.13) is the best we can do, for, although it distorts distances, it does preserve angles.

The Codazzi and Gauss equations are rather opaque, to say the least. We obtained the convenient equation ($*$) for the Gaussian curvature from the Gauss equations. To give a bit more insight into the meaning of the Codazzi equations, we have the following

**Lemma 3.3.** *Suppose* $\mathbf{x}$ *is a parametrization for which the $u$- and $v$-curves are lines of curvature, with respective principal curvatures $k_1$ and $k_2$. Then we have*

$$(\star) \qquad (k_1)_v = \frac{E_v}{2E}(k_2 - k_1) \qquad \text{and} \qquad (k_2)_u = \frac{G_u}{2G}(k_1 - k_2).$$