Proof. By Exercise 2.2.1, $\ell = k_1 E$, $n = k_2 G$, and $F = m = 0$. By the first Codazzi equation and the equations (‡) on p. 58, we have

$$(k_1)_v E + k_1 E_v = \ell_v = k_1 E \Gamma^u_{uv} - k_2 G \Gamma^v_{uu} = \tfrac{1}{2} E_v (k_1 + k_2),$$

and so

$$(k_1)_v = \frac{E_v}{2E}(k_2 - k_1).$$

The other formula follows similarly from the second Codazzi equation. □

Let's now apply the Codazzi equations to prove a rather striking result about the general surface with $K = 0$ everywhere.

Proposition 3.4. *Suppose M is a flat surface with no planar points. Then M is a ruled surface whose tangent plane is constant along the rulings.*

Proof. Since M has no planar points, we can choose $k_1 = 0$ and $k_2 \neq 0$ everywhere. Then by Theorem 3.3 of the Appendix, there is a local parametrization of M so that the u-curves are the first lines of curvature and the v-curves are the second lines of curvature. This means first of all that $F = m = 0$. (See Exercise 2.2.1.) Now, since $k_1 = 0$, for any $P \in M$ we have $S_P(\mathbf{x}_u) = \mathbf{0}$, and so $\mathbf{n}_u = \mathbf{0}$ everywhere and $\mathbf{n}$ is constant along the u-curves. We also observe that $\ell = \mathrm{II}(\mathbf{x}_u, \mathbf{x}_u) = -S_P(\mathbf{x}_u) \cdot \mathbf{x}_u = 0$.

We now want to show that the u-curves are in fact *lines*. Since $k_1 = 0$ everywhere, $(k_1)_v = 0$ and, since $k_2 \neq k_1$, we infer from Lemma 3.3 that $E_v = 0$. From the equations (‡) it now follows that $\Gamma^v_{uu} = 0$. Thus,

$$\mathbf{x}_{uu} = \Gamma^u_{uu}\mathbf{x}_u + \Gamma^v_{uu}\mathbf{x}_v + \ell\mathbf{n} = \Gamma^u_{uu}\mathbf{x}_u$$

is just a multiple of $\mathbf{x}_u$. Thus, the tangent vector $\mathbf{x}_u$ never changes direction as we move along the u-curves, and this means that the u-curves must be lines. In conclusion, we have a ruled surface whose tangent plane is constant along rulings. □

Remark. Flat ruled surfaces are often called *developable*. (See Exercise 10 and Exercise 2.1.12.) The terminology comes from the fact that they can be rolled out—or "developed"—onto a plane.

Next we prove a striking *global* result about compact surfaces. (Recall that a subset of $\mathbb{R}^3$ is *compact* if it is closed and bounded. The salient feature of compact sets is the maximum value theorem: A continuous real-valued function on a compact set achieves its maximum and minimum values.) We begin with a straightforward

Proposition 3.5. *Suppose $M \subset \mathbb{R}^3$ is a compact surface. Then there is a point $P \in M$ with $K(P) > 0$.*

Proof. Because M is compact, the continuous function $f(\mathbf{x}) = \|\mathbf{x}\|$ achieves its maximum at some point of M, and so there is a point $P \in M$ farthest from the origin (which may or may not be inside M), as indicated in Figure 3.2. Let $f(P) = R$. As Exercise 1.2.7 shows, the curvature of any curve $\boldsymbol{\alpha} \subset M$ at P is at least $1/R$. Applying this to any normal section of M at P and choosing the unit normal $\mathbf{n}$ to be inward-pointing, we deduce that every normal curvature of M at P is at least $1/R$. It follows that $K(P) \geq 1/R^2 > 0$. (That is, M is at least as curved at P as the circumscribed sphere of radius R tangent to M at P.) □

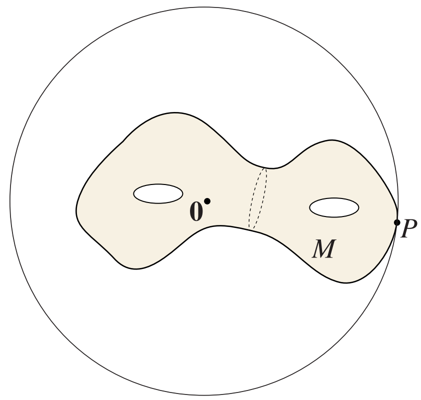

FIGURE 3.2

The reader is asked in Exercise 19 to find surfaces of revolution of constant curvature. There are, interestingly, many nonobvious examples. However, if we restrict ourselves to smooth, compact surfaces, we have the following beautiful

Theorem 3.6 (Liebmann)**.** *If M is a smooth, compact surface of constant Gaussian curvature K, then $K > 0$ and M must be a sphere of radius $1/\sqrt{K}$.*

We will need the following

Lemma 3.7 (Hilbert)**.** *Suppose P is not an umbilic point and $k_1(P) > k_2(P)$. Suppose k_1 has a local maximum at P and k_2 has a local minimum at P. Then $K(P) \le 0$.*

Proof. We work in a "principal" coordinate parametrization[7] near P, so that the u-curves are lines of curvature with principal curvature k_1 and the v-curves are lines of curvature with principal curvature k_2. Since $k_1 \ne k_2$ and $(k_1)_v = (k_2)_u = 0$ at P, it follows from Lemma 3.3 that $E_v = G_u = 0$ at P.

Differentiating the equations ($\star$), and remembering that $(k_1)_u = (k_2)_v = 0$ at P as well, we have *at* P:

$$(k_1)_{vv} = \frac{E_{vv}}{2E}(k_2 - k_1) \le 0 \qquad \text{(because } k_1 \text{ has a local maximum at } P\text{)}$$
$$(k_2)_{uu} = \frac{G_{uu}}{2G}(k_1 - k_2) \ge 0 \qquad \text{(because } k_2 \text{ has a local minimum at } P\text{),}$$

and so $E_{vv} \ge 0$ and $G_{uu} \ge 0$ at P. Using the equation ($*$) for the Gaussian curvature on p. 60, we see similarly that *at* P

$$K = -\frac{1}{2EG}\big(E_{vv} + G_{uu}\big),$$

as all the remaining terms involve E_v and G_u. So we conclude that $K(P) \le 0$, as desired. □

Proof of Theorem 3.6. By Proposition 3.5, there is a point where M is positively curved, and since the Gaussian curvature is constant, we must have $K > 0$. If every point is umbilic, then by Exercise 2.2.14, we know that M is a sphere. If there is some non-umbilic point, the larger principal curvature, k_1, achieves its maximum value at some point P because M is compact. Then, since $K = k_1 k_2$ is constant, the function $k_2 = K/k_1$ must achieve its minimum at P. Since P is necessarily a non-umbilic point (why?), it follows from Lemma 3.7 that $K(P) \le 0$, which is a contradiction. □

[7] Since locally there are no umbilic points, the existence of such a parametrization is an immediate consequence of Theorem 3.3 of the Appendix.

Remark. H. Hopf proved a stronger result, which requires techniques from complex analysis: If M is a compact surface topologically equivalent to a sphere and having constant *mean* curvature, then M must be a sphere.

We conclude this section with the analogue of Theorem 3.1 of Chapter 1.

Theorem 3.8 (Fundamental Theorem of Surface Theory)**.** Uniqueness: *Two parametrized surfaces* $\mathbf{x}, \mathbf{x}^*: U \to \mathbb{R}^3$ *are congruent (i.e., differ by a rigid motion) if and only if* $\mathrm{I} = \mathrm{I}^*$ *and* $\mathrm{II} = \pm\mathrm{II}^*$. Existence: *Moreover, given differentiable functions* E, F, G, ℓ, m, *and* n *with* $E > 0$ *and* $EG - F^2 > 0$ *and satisfying the Codazzi and Gauss equations, there exists (locally) a parametrized surface* $\mathbf{x}(u, v)$ *with the respective* I *and* II.

Proof. The existence statement requires some theorems from partial differential equations beyond our reach at this stage. The uniqueness statement, however, is much like the proof of Theorem 3.1 of Chapter 1. (The main technical difference is that we no longer are lucky enough to be working with an *orthonormal* basis at each point, as we were with the Frenet frame.)

First, suppose $\mathbf{x}^* = \Psi \circ \mathbf{x}$ for some rigid motion $\Psi: \mathbb{R}^3 \to \mathbb{R}^3$ (i.e., $\Psi(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ for some $\mathbf{b} \in \mathbb{R}^3$ and some 3×3 orthogonal matrix A). Since a translation doesn't change partial derivatives, we may assume that $\mathbf{b} = \mathbf{0}$. Now, since orthogonal matrices preserve length and dot product, we have $E^* = \|\mathbf{x}_u^*\|^2 = \|A\mathbf{x}_u\|^2 = \|\mathbf{x}_u\|^2 = E$, etc., so $\mathrm{I} = \mathrm{I}^*$. If $\det A > 0$, then $\mathbf{n}^* = A\mathbf{n}$, whereas if $\det A < 0$, then $\mathbf{n}^* = -A\mathbf{n}$. Thus, $\ell^* = \mathbf{x}_{uu}^* \cdot \mathbf{n}^* = A\mathbf{x}_{uu} \cdot (\pm A\mathbf{n}) = \pm\ell$, the positive sign holding when $\det A > 0$ and the negative when $\det A < 0$. Thus, $\mathrm{II}^* = \mathrm{II}$ if $\det A > 0$ and $\mathrm{II}^* = -\mathrm{II}$ if $\det A < 0$.

Conversely, suppose $\mathrm{I} = \mathrm{I}^*$ and $\mathrm{II} = \pm\mathrm{II}^*$. By composing $\mathbf{x}^*$ with a reflection, if necessary, we may assume that $\mathrm{II} = \mathrm{II}^*$. Now we need the following

Lemma 3.9. *Suppose* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\alpha}^*$ *are smooth functions on* $[0, b]$, $\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3$ *and* $\mathbf{v}_1^*\mathbf{v}_2^*\mathbf{v}_3^*$ *are smoothly varying bases for* $\mathbb{R}^3$, *also defined on* $[0, b]$, *so that*

$$\mathbf{v}_i(t) \cdot \mathbf{v}_j(t) = \mathbf{v}_i^*(t) \cdot \mathbf{v}_j^*(t) = g_{ij}(t), \qquad i, j = 1, 2, 3,$$

$$\boldsymbol{\alpha}'(t) = \sum_{i=1}^{3} p_i(t)\mathbf{v}_i(t) \qquad \text{and} \qquad \boldsymbol{\alpha}^{*\prime}(t) = \sum_{i=1}^{3} p_i(t)\mathbf{v}_i^*(t),$$
$$\mathbf{v}_j'(t) = \sum_{i=1}^{3} q_{ij}\mathbf{v}_i(t) \qquad \text{and} \qquad \mathbf{v}_j^{*\prime}(t) = \sum_{i=1}^{3} q_{ij}\mathbf{v}_i^*(t), \quad j = 1, 2, 3.$$

(Note that the coefficient functions p_i *and* q_{ij} *are the same for both the starred and unstarred equations.) If* $\boldsymbol{\alpha}(0) = \boldsymbol{\alpha}^*(0)$ *and* $\mathbf{v}_i(0) = \mathbf{v}_i^*(0)$, $i = 1, 2, 3$, *then* $\boldsymbol{\alpha}(t) = \boldsymbol{\alpha}^*(t)$ *and* $\mathbf{v}_i(t) = \mathbf{v}_i^*(t)$ *for all* $t \in [0, b]$, $i = 1, 2, 3$.

Fix a point $\mathbf{u}_0 \in U$. By composing $\mathbf{x}^*$ with a rigid motion, we may assume that *at* $\mathbf{u}_0$ we have $\mathbf{x} = \mathbf{x}^*$, $\mathbf{x}_u = \mathbf{x}_u^*$, $\mathbf{x}_v = \mathbf{x}_v^*$, and $\mathbf{n} = \mathbf{n}^*$ (why?). Choose an arbitrary $\mathbf{u}_1 \in U$, and join $\mathbf{u}_0$ to $\mathbf{u}_1$ by a path $\mathbf{u}(t)$, $t \in [0, b]$, and apply the lemma with $\boldsymbol{\alpha} = \mathbf{x} \circ \mathbf{u}$, $\mathbf{v}_1 = \mathbf{x}_u \circ \mathbf{u}$, $\mathbf{v}_2 = \mathbf{x}_v \circ \mathbf{u}$, $\mathbf{v}_3 = \mathbf{n} \circ \mathbf{u}$, $p_i = u_i'$, and the q_{ij} prescribed by the equations (†) and (††). Since $\mathrm{I} = \mathrm{I}^*$ and $\mathrm{II} = \mathrm{II}^*$, the same equations hold for $\boldsymbol{\alpha}^* = \mathbf{x}^* \circ \mathbf{u}$, and so $\mathbf{x}(\mathbf{u}_1) = \mathbf{x}^*(\mathbf{u}_1)$ as desired. That is, the two parametrized surfaces are identical. □