usual, work with a parametrization where the u-curves are lines of curvature with principal curvature k_1 and the v-curves are lines of curvature with principal curvature k_2. Use Lemma 3.3 to show that the u-curves have curvature $|k_1|$ and are planar. Then define $\boldsymbol{\alpha}$ appropriately and check that it is a regular curve.)

17. If M is a surface with both principal curvatures constant, prove that M is (a subset of) either a sphere, a plane, or a right circular cylinder. (Hint: See Exercise 2.2.14, Proposition 3.4, and Exercise 16.)

18. Consider the parametrized surfaces

$$\mathbf{x}(u,v) = (-\cosh u \sin v, \cosh u \cos v, u) \qquad \text{(a catenoid)}$$
$$\mathbf{y}(u,v) = (u \cos v, u \sin v, v) \qquad \text{(a helicoid)}.$$

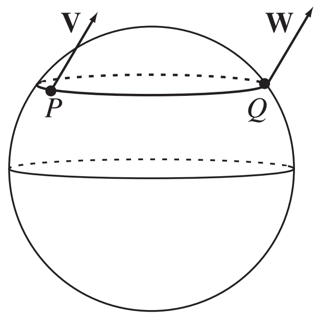
   a. Compute the first and second fundamental forms of both surfaces, and check that both surfaces are minimal.
   b. Find the asymptotic curves on both surfaces.
   c. Show that we can locally reparametrize the helicoid in such a way as to make the first fundamental forms of the two surfaces agree; this means that the two surfaces are locally isometric. (Hint: See p. 39. Replace u with $\sinh u$ in the parametrization of the helicoid. Why is this legitimate?)
   d. Why are they not globally isometric?
   e. (for the student who's seen a bit of complex variables) As a hint to what's going on here, let $z = u + iv$ and $\mathbf{Z} = \mathbf{x} + i\mathbf{y}$, and check that, continuing to use the substitution from part c, $\mathbf{Z} = (\sin iz, \cos iz, z)$. Understand now how one can obtain a one-parameter family of isometric surfaces interpolating between the helicoid and the catenoid.

19. Find all the surfaces of revolution of constant curvature
   a. $K = 0$
   b. $K = 1$
   c. $K = -1$

   (Hint: There are more than you might suspect. But your answers will involve integrals you cannot express in terms of elementary functions.)

4. Covariant Differentiation, Parallel Translation, and Geodesics

Now we turn to the "intrinsic" geometry of a surface, i.e., the geometry that can be observed by an inhabitant (for example, a very thin ant) of the surface, who can only perceive what happens along (or, say, tangential to) the surface. Anyone who has studied Euclidean geometry knows how important the notion of *parallelism* is (and classical non-Euclidean geometry arises when one removes Euclid's parallel postulate, which stipulates that given any line L in the plane and any point P not lying on L, there is a unique line through P parallel to L). It seems quite intuitive to say that, working just in $\mathbb{R}^3$, two vectors $\mathbf{V}$ (thought of as being "tangent at P") and $\mathbf{W}$ (thought of as being "tangent at Q") are parallel provided that we obtain $\mathbf{W}$ when we move $\mathbf{V}$ "parallel to itself" from P to Q; in other words, if $\mathbf{W} = \mathbf{V}$. But what would an inhabitant of the sphere say? How should he compare a tangent vector at one point of the sphere to a tangent vector



Are **V** and **W** parallel?

FIGURE 4.1

at another and determine if they're "parallel"? (See Figure 4.1.) Perhaps a better question is this: Given a curve $\boldsymbol{\alpha}$ on the surface and a vector field $\mathbf{X}$ defined along $\boldsymbol{\alpha}$, should we say $\mathbf{X}$ is parallel if it has zero derivative along $\boldsymbol{\alpha}$?

We already know how an inhabitant differentiates a scalar function $f: M \to \mathbb{R}$, by considering the directional derivative $D_{\mathbf{V}} f$ for any tangent vector $\mathbf{V} \in T_P M$. We now begin with a

Definition. We say a function $\mathbf{X}: M \to \mathbb{R}^3$ is a *vector field* on M if

(1) $\mathbf{X}(P) \in T_P M$ for every $P \in M$, and

(2) for any parametrization $\mathbf{x}: U \to M$, the function $\mathbf{X} \circ \mathbf{x}: U \to \mathbb{R}^3$ is (continuously) differentiable.

Now, we can differentiate a vector field $\mathbf{X}$ on M in the customary fashion: If $\mathbf{V} \in T_P M$, we choose a curve $\boldsymbol{\alpha}$ with $\boldsymbol{\alpha}(0) = P$ and $\boldsymbol{\alpha}'(0) = \mathbf{V}$ and set $D_{\mathbf{V}}\mathbf{X} = (\mathbf{X} \circ \boldsymbol{\alpha})'(0)$. (As usual, the chain rule tells us this is well-defined.) But the inhabitant of the surface can only see that portion of this vector lying in the tangent plane. This brings us to the

Definition. Given a vector field $\mathbf{X}$ and $\mathbf{V} \in T_P M$, we define the *covariant derivative*

$$\begin{aligned} \nabla_{\mathbf{V}}\mathbf{X} = (D_{\mathbf{V}}\mathbf{X})^{\parallel} &= \text{the projection of } D_{\mathbf{V}}\mathbf{X} \text{ onto } T_P M \\ &= D_{\mathbf{V}}\mathbf{X} - (D_{\mathbf{V}}\mathbf{X} \cdot \mathbf{n})\mathbf{n}. \end{aligned}$$

Given a curve $\boldsymbol{\alpha}$ in M, we say the vector field $\mathbf{X}$ is *covariant constant* or *parallel* along $\boldsymbol{\alpha}$ if $\nabla_{\boldsymbol{\alpha}'(t)}\mathbf{X} = \mathbf{0}$ for all t. (This means that $D_{\boldsymbol{\alpha}'(t)}\mathbf{X} = (\mathbf{X} \circ \boldsymbol{\alpha})'(t)$ is a multiple of the normal vector $\mathbf{n}(\boldsymbol{\alpha}(t))$.)

Example 1. Let M be a sphere and let $\boldsymbol{\alpha}$ be a great circle in M. The derivative of the unit tangent vector of $\boldsymbol{\alpha}$ points towards the center of the circle, which is in this case the center of the sphere, and thus is completely normal to the sphere. Therefore, the unit tangent vector field of $\boldsymbol{\alpha}$ is parallel along $\boldsymbol{\alpha}$. Observe that the constant vector field $(0, 0, 1)$ *is* parallel along the equator $z = 0$ of a sphere centered at the origin. Is this true of any other constant vector field? ∇

Example 2. A fundamental example requires that we revisit the Christoffel symbols. Given a parametrized surface $\mathbf{x}: U \to M$, we have

$$\begin{aligned} \nabla_{\mathbf{x}_u}\mathbf{x}_u &= (\mathbf{x}_{uu})^{\parallel} = \Gamma_{uu}^{u}\mathbf{x}_u + \Gamma_{uu}^{v}\mathbf{x}_v \\ \nabla_{\mathbf{x}_v}\mathbf{x}_u &= (\mathbf{x}_{uv})^{\parallel} = \Gamma_{uv}^{u}\mathbf{x}_u + \Gamma_{uv}^{v}\mathbf{x}_v = \nabla_{\mathbf{x}_u}\mathbf{x}_v, \quad \text{and} \end{aligned}$$

$$\nabla_{\mathbf{x}_v}\mathbf{x}_v = (\mathbf{x}_{vv})^{\|} = \Gamma^{u}_{vv}\mathbf{x}_u + \Gamma^{v}_{vv}\mathbf{x}_v. \quad \nabla$$

The first result we prove is the following

Proposition 4.1. *Let I be an interval in $\mathbb{R}$ with $0 \in I$. Given a curve $\boldsymbol{\alpha}\colon I \to M$ with $\boldsymbol{\alpha}(0) = P$ and $\mathbf{X}_0 \in T_P M$, there is a unique parallel vector field $\mathbf{X}$ defined along $\boldsymbol{\alpha}$ with $\mathbf{X}(P) = \mathbf{X}_0$.*

Proof. Assuming $\boldsymbol{\alpha}$ lies in a parametrized portion $\mathbf{x}\colon U \to M$, set $\boldsymbol{\alpha}(t) = \mathbf{x}(u(t), v(t))$ and write $\mathbf{X}(\boldsymbol{\alpha}(t)) = a(t)\mathbf{x}_u(u(t), v(t)) + b(t)\mathbf{x}_v(u(t), v(t))$. Then $\boldsymbol{\alpha}'(t) = u'(t)\mathbf{x}_u + v'(t)\mathbf{x}_v$ (where the the cumbersome argument $(u(t), v(t))$ is understood). So, by the product rule and chain rule, we have

$$\begin{aligned}
\nabla_{\boldsymbol{\alpha}'(t)}\mathbf{X} &= \big((\mathbf{X}\circ\boldsymbol{\alpha})'(t)\big)^{\|} = \left(\frac{d}{dt}\big(a(t)\mathbf{x}_u(u(t), v(t)) + b(t)\mathbf{x}_v(u(t), v(t))\big)\right)^{\|} \\
&= a'(t)\mathbf{x}_u + b'(t)\mathbf{x}_v + a(t)\left(\frac{d}{dt}\mathbf{x}_u(u(t), v(t))\right)^{\|} + b(t)\left(\frac{d}{dt}\mathbf{x}_v(u(t), v(t))\right)^{\|} \\
&= a'(t)\mathbf{x}_u + b'(t)\mathbf{x}_v + a(t)\big(u'(t)\mathbf{x}_{uu} + v'(t)\mathbf{x}_{uv}\big)^{\|} + b(t)\big(u'(t)\mathbf{x}_{vu} + v'(t)\mathbf{x}_{vv}\big)^{\|} \\
&= a'(t)\mathbf{x}_u + b'(t)\mathbf{x}_v + a(t)\big(u'(t)(\Gamma^{u}_{uu}\mathbf{x}_u + \Gamma^{v}_{uu}\mathbf{x}_v) + v'(t)(\Gamma^{u}_{uv}\mathbf{x}_u + \Gamma^{v}_{uv}\mathbf{x}_v)\big) \\
&\qquad + b(t)\big(u'(t)(\Gamma^{u}_{vu}\mathbf{x}_u + \Gamma^{v}_{vu}\mathbf{x}_v) + v'(t)(\Gamma^{u}_{vv}\mathbf{x}_u + \Gamma^{v}_{vv}\mathbf{x}_v)\big) \\
&= \big(a'(t) + a(t)(\Gamma^{u}_{uu}u'(t) + \Gamma^{u}_{uv}v'(t)) + b(t)(\Gamma^{u}_{vu}u'(t) + \Gamma^{u}_{vv}v'(t))\big)\mathbf{x}_u \\
&\qquad + \big(b'(t) + a(t)(\Gamma^{v}_{uu}u'(t) + \Gamma^{v}_{uv}v'(t)) + b(t)(\Gamma^{v}_{vu}u'(t) + \Gamma^{v}_{vv}v'(t))\big)\mathbf{x}_v.
\end{aligned}$$

Thus, to say $\mathbf{X}$ is parallel along the curve $\boldsymbol{\alpha}$ is to say that $a(t)$ and $b(t)$ are solutions of the linear system of first order differential equations

$$(\clubsuit) \qquad \begin{aligned}
a'(t) + a(t)(\Gamma^{u}_{uu}u'(t) + \Gamma^{u}_{uv}v'(t)) + b(t)(\Gamma^{u}_{vu}u'(t) + \Gamma^{u}_{vv}v'(t)) &= 0 \\
b'(t) + a(t)(\Gamma^{v}_{uu}u'(t) + \Gamma^{v}_{uv}v'(t)) + b(t)(\Gamma^{v}_{vu}u'(t) + \Gamma^{v}_{vv}v'(t)) &= 0.
\end{aligned}$$

By Theorem 3.2 of the Appendix, this system has a unique solution on I once we specify $a(0)$ and $b(0)$, and hence we obtain a unique parallel vector field $\mathbf{X}$ with $\mathbf{X}(P) = \mathbf{X}_0$. $\square$

Definition. If $\boldsymbol{\alpha}$ is a path from P to Q, we refer to $\mathbf{X}(Q)$ as the *parallel translate* of $\mathbf{X}(P) = \mathbf{X}_0 \in T_P M$ along $\boldsymbol{\alpha}$, or the result of *parallel translation* along $\boldsymbol{\alpha}$.

Remark. The system of differential equations ($\clubsuit$) that defines parallel translation shows that it is "intrinsic," i.e., depends only on the first fundamental form of M, despite our original extrinsic definition. In particular, parallel translation in locally isometric surfaces will be identical.

Example 3. Fix a latitude circle $u = u_0$ ($u_0 \neq 0, \pi$) on the unit sphere (see Example 1(d) on p. 37) and let's calculate the effect of parallel-translating the vector $\mathbf{X}_0 = \mathbf{x}_v$ starting at the point P given by $u = u_0$, $v = 0$, once around the circle, counterclockwise. We parametrize the curve by $u(t) = u_0$, $v(t) = t$, $0 \le t \le 2\pi$. Using our computation of the Christoffel symbols of the sphere in Example 1 or 2 of Section 3, we obtain from ($\clubsuit$) the differential equations

$$\begin{aligned}
a'(t) &= \sin u_0 \cos u_0 b(t), \qquad & a(0) &= 0 \\
b'(t) &= -\cot u_0 a(t), & b(0) &= 1.
\end{aligned}$$

We solve this system by differentiating the second equation again and substituting the first:

$$b''(t) = -\cot u_0 a'(t) = -\cos^2 u_0 b(t), \qquad b(0) = 1.$$

Recalling that every solution of the differential equation $y''(t) + k^2 y(t) = 0$ is of the form $y(t) = c_1 \cos(kt) + c_2 \sin(kt)$, $c_1, c_2 \in \mathbb{R}$, we see that the solution is

$$a(t) = \sin u_0 \sin\big((\cos u_0)t\big), \qquad b(t) = \cos\big((\cos u_0)t\big).$$

Note that $\|\mathbf{X}(\boldsymbol{\alpha}(t))\|^2 = Ea(t)^2 + 2Fa(t)b(t) + Gb(t)^2 = \sin^2 u_0$ for all t. That is, the original vector $\mathbf{X}_0$ rotates as we parallel translate it around the latitude circle, and its length is preserved. As we see in Figure 4.2, the vector rotates clockwise as we proceed around the latitude circle (in the upper hemisphere). But

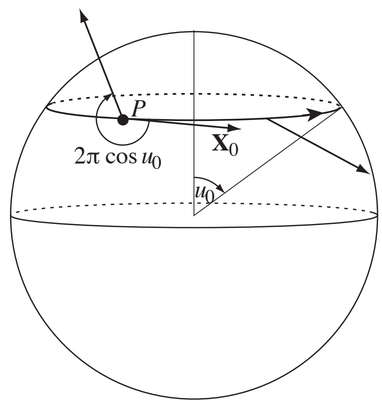


FIGURE 4.2

this makes sense: If we just take the covariant derivative of the vector field tangent to the circle, it points upwards (cf. Figure 3.1), so the vector field must rotate clockwise to counteract that effect in order to remain parallel. Since $b(2\pi) = \cos(2\pi \cos u_0)$, we see that the vector turns through an angle of $-2\pi \cos u_0$. ▽

Example 4 (Foucault pendulum)**.** Foucault observed in 1851 that the swing plane of a pendulum located on the latitude circle $u = u_0$ precesses with a period of $T = 24/\cos u_0$ hours. We can use the result of Example 3 to explain this. We imagine the earth as fixed and "transport" the swinging pendulum once around the circle in 24 hours. If we make the pendulum very long and the swing rather short, the motion will be "essentially" tangential to the surface of the earth. If we move slowly around the circle, the forces will be "essentially" normal to the sphere: In particular, letting R denote the radius of the earth (approximately 3960 mi), the tangential component of the centripetal acceleration is (cf. Figure 3.1)

$$(R \sin u_0) \cos u_0 \left(\frac{2\pi}{24}\right)^2 \leq \frac{2\pi^2 R}{24^2} \approx 135.7 \text{ mi/hr}^2 \approx 0.0553 \text{ ft/sec}^2 \approx 0.17\%g.$$

Thus, the "swing vector field" is, for all practical purposes, parallel along the curve. Therefore, it turns through an angle of $2\pi \cos u_0$ in one trip around the circle, so it takes $\dfrac{2\pi}{(2\pi \cos u_0)/24} = \dfrac{24}{\cos u_0}$ hours to return to its original swing plane. ▽

Our experience in Example 3 suggests the following