Proposition 4.2. *Parallel translation preserves lengths and angles. That is, if* $\mathbf{X}$ *and* $\mathbf{Y}$ *are parallel vector fields along a curve* $\boldsymbol{\alpha}$ *from* P *to* Q, *then* $\|\mathbf{X}(P)\| = \|\mathbf{X}(Q)\|$ *and the angle between* $\mathbf{X}(P)$ *and* $\mathbf{Y}(P)$ *equals the angle between* $\mathbf{X}(Q)$ *and* $\mathbf{Y}(Q)$ *(assuming these are nonzero vectors).*

Proof. Consider $f(t) = \mathbf{X}(\boldsymbol{\alpha}(t)) \cdot \mathbf{Y}(\boldsymbol{\alpha}(t))$. Then

$$\begin{aligned} f'(t) &= (\mathbf{X}\circ\boldsymbol{\alpha})'(t) \cdot (\mathbf{Y}\circ\boldsymbol{\alpha})(t) + (\mathbf{X}\circ\boldsymbol{\alpha})(t) \cdot (\mathbf{Y}\circ\boldsymbol{\alpha})'(t) \\ &= D_{\boldsymbol{\alpha}'(t)}\mathbf{X} \cdot \mathbf{Y} + \mathbf{X} \cdot D_{\boldsymbol{\alpha}'(t)}\mathbf{Y} \overset{(1)}{=} \nabla_{\boldsymbol{\alpha}'(t)}\mathbf{X} \cdot \mathbf{Y} + \mathbf{X} \cdot \nabla_{\boldsymbol{\alpha}'(t)}\mathbf{Y} \overset{(2)}{=} 0. \end{aligned}$$

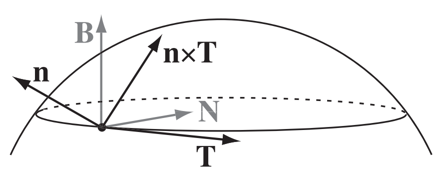
Note that equality (1) holds because $\mathbf{X}$ and $\mathbf{Y}$ are tangent to M and hence their dot product with any vector normal to the surface is 0. Equality (2) holds because $\mathbf{X}$ and $\mathbf{Y}$ are assumed parallel along $\boldsymbol{\alpha}$. It follows that the dot product $\mathbf{X} \cdot \mathbf{Y}$ remains constant along $\boldsymbol{\alpha}$. Taking $\mathbf{Y} = \mathbf{X}$, we infer that $\|\mathbf{X}\|$ (and similarly $\|\mathbf{Y}\|$) is constant. Knowing that, using the famous formula $\cos\theta = \mathbf{X} \cdot \mathbf{Y}/\|\mathbf{X}\|\|\mathbf{Y}\|$ for the angle θ between $\mathbf{X}$ and $\mathbf{Y}$, we infer that the angle remains constant. ☐

Now we change gears somewhat. We saw in Exercise 1.1.8 that the shortest path joining two points in $\mathbb{R}^3$ is a line segment and in Exercise 1.3.1 that the shortest path joining two points on the unit sphere is a great circle. One characterization of the line segment is that it never changes direction, so that its unit tangent vector is parallel (so no distance is wasted by turning). (What about the sphere?) It seems plausible that the mythical inhabitant of our general surface M might try to travel from one point to another in M, *staying in* M, by similarly not turning; that is, so that his unit tangent vector field is parallel along his path. Physically, this means that if he travels at constant speed, any acceleration should be normal to the surface. This leads us to the following

Definition. We say a parametrized curve $\boldsymbol{\alpha}$ in a surface M is a *geodesic* if its tangent vector is parallel along the curve, i.e., if $\nabla_{\boldsymbol{\alpha}'}\boldsymbol{\alpha}' = 0$.

Recall that since parallel translation preserves lengths, $\boldsymbol{\alpha}$ must have constant speed, although it may not be arclength-parametrized. In general, we refer to an unparametrized curve as a geodesic if its arclength parametrization is in fact a geodesic.

In general, given any arclength-parametrized curve $\boldsymbol{\alpha}$ lying on M, we defined its normal curvature at the end of Section 2. Instead of using the Frenet frame, it is natural to consider the *Darboux frame* for $\boldsymbol{\alpha}$, which takes into account the fact that $\boldsymbol{\alpha}$ lies on the surface M. (Both are illustrated in Figure 4.3.) We take



The Frenet and Darboux frames

FIGURE 4.3

the right-handed orthonormal basis $\{\mathbf{T}, \mathbf{n}\times\mathbf{T}, \mathbf{n}\}$; note that the first two vectors give a basis for $T_P M$. We can decompose the curvature vector

$$\kappa\mathbf{N} = \big(\underbrace{\kappa\mathbf{N}\cdot(\mathbf{n}\times\mathbf{T})}_{\kappa_g}\big)(\mathbf{n}\times\mathbf{T}) + \big(\underbrace{\kappa\mathbf{N}\cdot\mathbf{n}}_{\kappa_n}\big)\mathbf{n}.$$

As we saw before, κ_n gives the *normal* component of the curvature vector; κ_g gives the *tangential* component of the curvature vector and is called the *geodesic curvature*. This terminology arises from the fact that $\boldsymbol{\alpha}$ is a geodesic if and only if its geodesic curvature vanishes. (When $\kappa = 0$, the principal normal is not defined, and we really should write $\boldsymbol{\alpha}''$ in the place of $\kappa\mathbf{N}$. If the acceleration vanishes at a point, then certainly its normal and tangential components are both $\mathbf{0}$.)

Example 5. We saw in Example 1 that every great circle on a sphere is a geodesic. Are there others? Let $\boldsymbol{\alpha}$ be a geodesic on a sphere centered at the origin. Since $\kappa_g = 0$, the acceleration vector $\boldsymbol{\alpha}''(s)$ must be a multiple of $\boldsymbol{\alpha}(s)$ for every s, and so $\boldsymbol{\alpha}'' \times \boldsymbol{\alpha} = \mathbf{0}$. Therefore $\boldsymbol{\alpha}' \times \boldsymbol{\alpha} = \mathbf{A}$ is a constant vector, so $\boldsymbol{\alpha}$ lies in the plane passing through the origin with normal vector $\mathbf{A}$. That is, $\boldsymbol{\alpha}$ is a great circle. ∇

Remark. We saw in Example 3 that a vector rotates clockwise at a constant rate as we parallel translate along the latitude circle of the sphere. If we think about the unit tangent vector $\mathbf{T}$ moving counterclockwise along this curve, its covariant derivative along the curve points up the sphere, as shown in Figure 4.4, i.e., "to the left." Thus, we must compensate by steering "to the right" in order to have no net turning (i.e., to

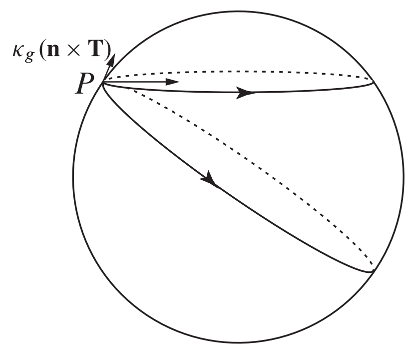


FIGURE 4.4

make the covariant derivative zero). Of course, this makes sense also because, according to Example 5, the geodesic that passes through P in the same direction heads "downhill," to the right.

Using the equations (♣), let's now give the equations for the curve $\boldsymbol{\alpha}(t) = \mathbf{x}(u(t), v(t))$ to be a geodesic. Since $\mathbf{X} = \boldsymbol{\alpha}'(t) = u'(t)\mathbf{x}_u + v'(t)\mathbf{x}_v$, we have $a(t) = u'(t)$ and $b(t) = v'(t)$, and the resulting equations are

(♣♣)
$$\begin{aligned} u''(t) + \Gamma_{uu}^{u}u'(t)^2 + 2\Gamma_{uv}^{u}u'(t)v'(t) + \Gamma_{vv}^{u}v'(t)^2 &= 0 \\ v''(t) + \Gamma_{uu}^{v}u'(t)^2 + 2\Gamma_{uv}^{v}u'(t)v'(t) + \Gamma_{vv}^{v}v'(t)^2 &= 0. \end{aligned}$$

The following result is a consequence of basic results on differential equations (see Theorem 3.1 of the Appendix).

Proposition 4.3. *Given a point $P \in M$ and $\mathbf{V} \in T_PM$, $\mathbf{V} \neq \mathbf{0}$, there exist $\varepsilon > 0$ and a* unique *geodesic $\boldsymbol{\alpha}\colon (-\varepsilon, \varepsilon) \to M$ with $\boldsymbol{\alpha}(0) = P$ and $\boldsymbol{\alpha}'(0) = \mathbf{V}$.*

Example 6. We now use the equations (♣♣) to solve for geodesics analytically in a few examples.

(a) Let $\mathbf{x}(u, v) = (u, v)$ be the obvious parametrization of the plane. Then all the Christoffel symbols vanish and the geodesics are the solutions of

$$u''(t) = v''(t) = 0,$$

so we get the lines $\boldsymbol{\alpha}(t) = (u(t), v(t)) = (a_1 t + b_1, a_2 t + b_2)$, as expected. Note that $\boldsymbol{\alpha}$ does in fact have constant speed.

(b) Using the standard spherical coordinate parametrization of the sphere, we obtain (see Example 1 or 2 of Section 3) the equations

$$(*) \qquad u''(t) - \sin u(t) \cos u(t) v'(t)^2 = 0 = v''(t) + 2 \cot u(t) u'(t) v'(t).$$

Well, one obvious set of solutions is to take $u(t) = t$, $v(t) = v_0$ (and these, indeed, give the great circles through the north pole). Integrating the second equation in $(*)$ we obtain $\ln v'(t) = -2 \ln \sin u(t) + \text{const}$, so

$$v'(t) = \frac{c}{\sin^2 u(t)}$$

for some constant c. Substituting this in the first equation in $(*)$ we find that

$$u''(t) - \frac{c^2 \cos u(t)}{\sin^3 u(t)} = 0;$$

multiplying both sides by $u'(t)$ (the "energy trick" from physics) and integrating, we get

$$u'(t)^2 = C^2 - \frac{c^2}{\sin^2 u(t)}, \quad \text{and so} \quad u'(t) = \pm\sqrt{C^2 - \frac{c^2}{\sin^2 u(t)}}$$

for some constant C. Switching to Leibniz notation for obvious reasons, we obtain

$$\frac{dv}{du} = \frac{v'(t)}{u'(t)} = \pm\frac{c \csc^2 u}{\sqrt{C^2 - c^2 \csc^2 u}}; \quad \text{thus, separating variables gives}$$

$$dv = \pm\frac{c \csc^2 u\, du}{\sqrt{C^2 - c^2 \csc^2 u}} = \pm\frac{c \csc^2 u\, du}{\sqrt{(C^2 - c^2) - c^2 \cot^2 u}}.$$

Now we make the substitution $c \cot u = \sqrt{C^2 - c^2} \sin w$; then we have

$$dv = \pm\frac{c \csc^2 u\, du}{\sqrt{(C^2 - c^2) - c^2 \cot^2 u}} = \mp dw,$$

and so, at long last, we have $w = \pm v + a$ for some constant a. Thus,

$$c \cot u = \sqrt{C^2 - c^2} \sin w = \sqrt{C^2 - c^2} \sin(\pm v + a) = \sqrt{C^2 - c^2}(\sin a \cos v \pm \cos a \sin v),$$

and so, finally, we have the equation

$$c \cos u + \sqrt{C^2 - c^2} \sin u (A \cos v + B \sin v) = 0,$$

which we should recognize as the equation of a great circle! (Here's a hint: This curve lies on the plane $\sqrt{C^2 - c^2}(Ax + By) + cz = 0$.) $\qquad \nabla$

We can now give a beautiful geometric description of the geodesics on a surface of revolution.