Proposition 4.4 (Clairaut's relation)**.** *The geodesics on a surface of revolution satisfy the equation*

$$(\diamondsuit) \qquad r\cos\phi = \text{const},$$

where r is the distance from the axis of revolution and ϕ is the angle between the geodesic and the parallel. Conversely, any (constant speed) curve satisfying ($\diamondsuit$) that is not *a parallel is a geodesic.*

Proof. For the surface of revolution parametrized as in Example 9 of Section 2, we have $E = 1$, $F = 0$, $G = f(u)^2$, $\Gamma^v_{uv} = \Gamma^v_{vu} = f'(u)/f(u)$, $\Gamma^u_{vv} = -f(u)f'(u)$, and all other Christoffel symbols are 0 (see Exercise 2.3.2d.). Then the system (♣♣) of differential equations becomes

$$(\dagger_1) \qquad u'' - ff'(v')^2 = 0$$

$$(\dagger_2) \qquad v'' + \frac{2f'}{f}u'v' = 0.$$

Rewriting the equation ($\dagger_2$) and integrating, we obtain

$$\begin{aligned} \frac{v''(t)}{v'(t)} &= -\frac{2f'(u(t))u'(t)}{f(u(t))} \\ \ln v'(t) &= -2\ln f(u(t)) + \text{const} \\ v'(t) &= \frac{c}{f(u(t))^2}, \end{aligned}$$

so along a geodesic the quantity $f(u)^2 v' = Gv'$ is constant. We recognize this as the dot product of the tangent vector of our geodesic with the vector $\mathbf{x}_v$, and so we infer that $\|\mathbf{x}_v\|\cos\phi = r\cos\phi$ is constant. (Recall that, by Proposition 4.2, the tangent vector of the geodesic has constant length.)

To this point we have seen that the equation ($\dagger_2$) is equivalent to the condition $r\cos\phi = \text{const}$, provided we assume $\|\boldsymbol{\alpha}'\|^2 = u'^2 + Gv'^2$ is constant as well. But if

$$u'(t)^2 + Gv'(t)^2 = u'(t)^2 + f(u(t))^2 v'(t)^2 = \text{const},$$

we differentiate and obtain

$$u'(t)u''(t) + f(u(t))^2 v'(t)v''(t) + f(u(t))f'(u(t))u'(t)v'(t)^2 = 0;$$

substituting for $v''(t)$ using ($\dagger_2$), we find

$$u'(t)\big(u''(t) - f(u(t))f'(u(t))v'(t)^2\big) = 0.$$

In other words, *provided* $u'(t) \neq 0$, a constant-speed curve satisfying ($\dagger_2$) satisfies ($\dagger_1$) as well. (See Exercise 6 for the case of the parallels.) □

Remark. We can give a simple physical interpretation of Clairaut's relation. Imagine a particle with mass 1 constrained to move along a surface. If no external forces are acting, then the particle moves along a geodesic and, moreover, angular momentum is conserved (because there are no torques). In the case of our surface of revolution, the vertical component of the angular momentum $\mathbf{L} = \boldsymbol{\alpha} \times \boldsymbol{\alpha}'$ is—surprise, surprise!—$f^2 v'$, which we've shown is constant. Perhaps some forces normal to the surface are required to keep the particle on the surface; then the particle still moves along a geodesic (why?). Moreover, since $(\boldsymbol{\alpha} \times \mathbf{n}) \cdot (0, 0, 1) = 0$, the resulting torques *still* have no vertical component.

Returning to our original motivation for geodesics, we now consider the following scenario. Choose $P \in M$ arbitrary and a geodesic $\boldsymbol{\gamma}$ through P, and draw a curve C_0 through P orthogonal to $\boldsymbol{\gamma}$. We now choose a parametrization $\mathbf{x}(u, v)$ so that $\mathbf{x}(0,0) = P$, the u-curves are geodesics orthogonal to C_0, and the v-curves are the orthogonal trajectories of the u-curves, as pictured in Figure 4.5. (It follows from Theorem

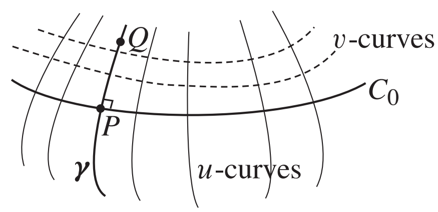

FIGURE 4.5

3.3 of the Appendix that we can do this on some neighborhood of P.)

In this parametrization we have $F = 0$ and $E = E(u)$ (see Exercise 13). Now, if $\boldsymbol{\alpha}(t) = \mathbf{x}(u(t), v(t))$, $a \le t \le b$, is any path from $P = \mathbf{x}(0,0)$ to $Q = \mathbf{x}(u_0, 0)$, we have

$$\begin{aligned}\text{length}(\boldsymbol{\alpha}) &= \int_a^b \sqrt{E(u(t))u'(t)^2 + G(u(t), v(t))v'(t)^2}\,dt \ge \int_a^b \sqrt{E(u(t))}|u'(t)|\,dt \\ &\ge \int_0^{u_0} \sqrt{E(u)}\,du,\end{aligned}$$

which is the length of the geodesic arc $\boldsymbol{\gamma}$ from P to Q. Thus, we have deduced the following.

Proposition 4.5. *For any point Q on $\boldsymbol{\gamma}$ contained in this parametrization, any path from P to Q* contained in this parametrization *is at least as long as the length of the geodesic segment. More colloquially, geodesics are* locally *distance-minimizing.*

Example 7. Why is Proposition 4.5 a local statement? Well, consider a great circle on a sphere, as shown in Figure 4.6. If we go more than halfway around, we obviously have not taken the shortest path. ∇

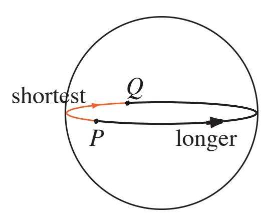

FIGURE 4.6

Remark. It turns out that any surface can be endowed with a *metric* (or *distance measure*) by defining the distance between any two points to be the infimum (usually, the minimum) of the lengths of all piecewise-$\mathcal{C}^1$ paths joining them. (Although the distance measure is different from the Euclidean distance as the surface sits in $\mathbb{R}^3$, the topology—notion of "neighborhood"—induced by this metric structure is the induced topology that the surface inherits as a subspace of $\mathbb{R}^3$.) It is a consequence of the Hopf-Rinow Theorem (see M. doCarmo, *Differential Geometry of Curves and Surfaces*, Prentice Hall, 1976, p. 333, or M. Spivak, *A*

Comprehensive Introduction to Differential Geometry, third edition, volume 1, Publish or Perish, Inc., 1999, p. 342) that in a surface in which every parametrized geodesic is defined for all time (a "complete" surface), every two points are in fact joined by a geodesic of least length. The proof of this result is quite tantalizing: To find the shortest path from P to Q, one walks around the "geodesic circle" of points a small distance from P and finds the point R on it closest to Q; one then proves that the unique geodesic emanating from P that passes through R must eventually pass through Q, and there can be no shorter path.

We referred earlier to two surfaces M and M^* as being globally isometric (e.g., in Example 6 in Section 1). We can now give the official definition: There should be a function $f: M \to M^*$ that establishes a one-to-one correspondence and preserves distance—for any $P, Q \in M$, the distance between P and Q in M should be equal to the distance between $f(P)$ and $f(Q)$ in M^*.

EXERCISES 2.4

1. Determine the result of parallel translating the vector $(0, 0, 1)$ once around the circle $x^2 + y^2 = a^2$, $z = 0$, on the right circular cylinder $x^2 + y^2 = a^2$.

2. Prove that $\kappa^2 = \kappa_g^2 + \kappa_n^2$.

3. Suppose $\boldsymbol{\alpha}$ is a non-arclength-parametrized curve. Using the formula $(**)$ on p. 14, prove that the velocity vector of $\boldsymbol{\alpha}$ is parallel along $\boldsymbol{\alpha}$ if and only if $\kappa_g = 0$ and $\upsilon' = 0$.

*4. Find the geodesic curvature κ_g of a latitude circle $u = u_0$ on the unit sphere (see Example 1(d) on p. 37)
   a. directly
   b. by applying the result of Exercise 2

5. Consider the right circular cone with vertex angle 2ϕ parametrized by

$$\mathbf{x}(u, v) = (u \tan\phi \cos v, u \tan\phi \sin v, u), \quad 0 < u \le u_0,\ 0 \le v \le 2\pi.$$

   Find the geodesic curvature κ_g of the circle $u = u_0$ by using trigonometric considerations. Check that your answer agrees with the curvature of the circle you get by unrolling the cone to form a "pacman" figure, as shown on the left in Figure 4.7. (For a proof that these curvatures should agree, see Exercise 2.1.10 and Exercise 3.1.7.)

6. Check that the parallel $u = u_0$ is a geodesic on the surface of revolution parametrized as in Proposition 4.4 if and only if $f'(u_0) = 0$. Give a geometric interpretation of and explanation for this result.

7. Use the equations (♣), as in Example 3, to determine through what angle a vector turns when it is parallel-translated once around the circle $u = u_0$ on the cone $\mathbf{x}(u, v) = (u\cos v, u \sin v, cu)$, $c \neq 0$. (See Exercise 2.3.2c.)

8. a. Prove that if the surfaces M and M^* are tangent along the curve C, parallel translation along C is the same in both surfaces.