CHAPTER 3

Surfaces: Further Topics

The first section is required reading, but the remaining sections of this chapter are independent of one another.

1. Holonomy and the Gauss-Bonnet Theorem

Let's now pursue the discussion of parallel translation that we began in Chapter 2. Let M be a surface and $\boldsymbol{\alpha}$ a closed curve in M. We begin by fixing a smoothly-varying orthonormal basis $\mathbf{e}_1, \mathbf{e}_2$ (a so-called *framing*) for the tangent planes of M in an open set of M containing $\boldsymbol{\alpha}$, as shown in Figure 1.1 below. Now

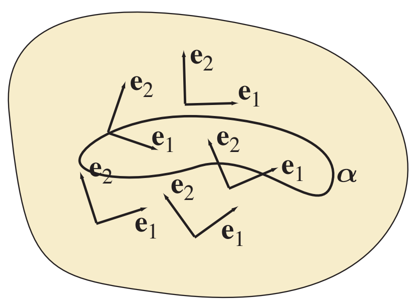

FIGURE 1.1

we make the following

Definition. Let $\boldsymbol{\alpha}$ be a *closed* curve in a surface M. The angle through which a vector turns *relative to the given framing* as we parallel translate it once around the curve $\boldsymbol{\alpha}$ is called the *holonomy*[1] around $\boldsymbol{\alpha}$.

For example, if we take a framing around $\boldsymbol{\alpha}$ by using the unit tangent vectors to $\boldsymbol{\alpha}$ as our vectors $\mathbf{e}_1$, then, by the definition of a geodesic, there there will be zero holonomy around a closed geodesic (why?). For another example, if we use the framing on (most of) the sphere given by the tangents to the lines of longitude and lines of latitude, the computation in Example 3 of Section 4 of Chapter 2 shows that the holonomy around a latitude circle $u = u_0$ of the unit sphere is $-2\pi \cos u_0$.

To make this more precise, for ease of understanding, let's work in an orthogonal parametrization[2] and define a framing by setting

$$\mathbf{e}_1 = \frac{\mathbf{x}_u}{\sqrt{E}} \qquad \text{and} \qquad \mathbf{e}_2 = \frac{\mathbf{x}_v}{\sqrt{G}}.$$

Since (much as in the case of curves) $\mathbf{e}_1$ and $\mathbf{e}_2$ give an orthonormal basis for the tangent space of our surface at each point, all the intrinsic curvature information (such as given by the Christoffel symbols)

[1] from *holo-+-nomy*, the study of the whole

[2] As usual, away from umbilic points, we can apply Theorem 3.3 of the Appendix to obtain a parametrization where the u- and v-curves are lines of curvature.

is encapsulated in knowing how $\mathbf{e}_1$ twists towards $\mathbf{e}_2$ as we move around the surface. In particular, if $\boldsymbol{\alpha}(t) = \mathbf{x}(u(t), v(t))$, $a \le t \le b$, is a parametrized curve, we can set

$$\phi_{12}(t) = \frac{d}{dt}\big(\mathbf{e}_1(u(t), v(t))\big) \cdot \mathbf{e}_2(u(t), v(t)),$$

which we may write more casually as $\mathbf{e}_1'(t) \cdot \mathbf{e}_2(t)$, with the understanding that everything must be done in terms of the parametrization. We emphasize that ϕ_{12} depends in an essential way on the *parametrized* curve $\boldsymbol{\alpha}$. Perhaps it's better, then, to write

$$\phi_{12} = \nabla_{\boldsymbol{\alpha}'}\mathbf{e}_1 \cdot \mathbf{e}_2.$$

Note, moreover, that the proof of Proposition 4.2 of Chapter 2 shows that $\nabla_{\boldsymbol{\alpha}'}\mathbf{e}_2 \cdot \mathbf{e}_1 = -\phi_{12}$ and $\nabla_{\boldsymbol{\alpha}'}\mathbf{e}_1 \cdot \mathbf{e}_1 = \nabla_{\boldsymbol{\alpha}'}\mathbf{e}_2 \cdot \mathbf{e}_2 = 0$. (Why?)

Remark. Although the notation seems cumbersome, it reminds us that ϕ_{12} is measuring how $\mathbf{e}_1$ *twists* towards $\mathbf{e}_2$ as we move along the curve $\boldsymbol{\alpha}$. This notation will fit in a more general context in Section 3.

Let's now derive an explicit formula for the function ϕ_{12}.

Proposition 1.1. *In an orthogonal parametrization with* $\mathbf{e}_1 = \mathbf{x}_u/\sqrt{E}$ *and* $\mathbf{e}_2 = \mathbf{x}_v/\sqrt{G}$, *we have* $\phi_{12} = \dfrac{1}{2\sqrt{EG}}\big(-E_v u' + G_u v'\big)$.

Proof. The key point is to take full advantage of the orthogonality of $\mathbf{x}_u$ and $\mathbf{x}_v$.

$$\begin{aligned}\phi_{12} &= \frac{d}{dt}\left(\frac{\mathbf{x}_u}{\sqrt{E}}\right) \cdot \frac{\mathbf{x}_v}{\sqrt{G}} \\ &= \frac{1}{\sqrt{EG}}\big(\mathbf{x}_{uu}u' + \mathbf{x}_{uv}v'\big) \cdot \mathbf{x}_v\end{aligned}$$

(since the term that would arise from differentiating $\sqrt{E}$ will involve $\mathbf{x}_u \cdot \mathbf{x}_v = 0$)

$$= \frac{1}{2\sqrt{EG}}\big(-E_v u' + G_u v'\big),$$

by the formulas (♠) on p. 58. □

Suppose now that $\boldsymbol{\alpha}$ is a *closed* curve and we are interested in the holonomy around $\boldsymbol{\alpha}$. If $\mathbf{e}_1$ happens to be parallel along $\boldsymbol{\alpha}$, then the holonomy will, of course, be 0. If not, let's consider $\mathbf{X}(t)$ to be the parallel translation of $\mathbf{e}_1$ along $\boldsymbol{\alpha}(t)$ and write $\mathbf{X}(t) = \cos\psi(t)\mathbf{e}_1 + \sin\psi(t)\mathbf{e}_2$, taking $\psi(0) = 0$. Then $\mathbf{X}$ is parallel along $\boldsymbol{\alpha}$ if and only if

$$\begin{aligned}\mathbf{0} = \nabla_{\boldsymbol{\alpha}'}\mathbf{X} &= \nabla_{\boldsymbol{\alpha}'}(\cos\psi\mathbf{e}_1 + \sin\psi\mathbf{e}_2) \\ &= \cos\psi\nabla_{\boldsymbol{\alpha}'}\mathbf{e}_1 + \sin\psi\nabla_{\boldsymbol{\alpha}'}\mathbf{e}_2 + (-\sin\psi\mathbf{e}_1 + \cos\psi\mathbf{e}_2)\psi' \\ &= \cos\psi\phi_{12}\mathbf{e}_2 - \sin\psi\phi_{12}\mathbf{e}_1 + (-\sin\psi\mathbf{e}_1 + \cos\psi\mathbf{e}_2)\psi' \\ &= (\phi_{12} + \psi')(-\sin\psi\mathbf{e}_1 + \cos\psi\mathbf{e}_2).\end{aligned}$$

Thus, $\mathbf{X}$ is parallel along $\boldsymbol{\alpha}$ if and only if $\psi'(t) = -\phi_{12}(t)$. We therefore conclude:

Proposition 1.2. *The holonomy around the closed curve C equals* $\Delta\psi = -\displaystyle\int_a^b \phi_{12}(t)\,dt$.

Remark. Note that the angle ψ is measured from $\mathbf{e}_1$ in the direction of $\mathbf{e}_2$. Whether the vector turns counterclockwise or clockwise from our external viewpoint depends on the orientation of the framing.

Example 1. Back to our example of the latitude circle $u = u_0$ on the unit sphere. Then $\mathbf{e}_1 = \mathbf{x}_u$ and $\mathbf{e}_2 = (1/\sin u)\mathbf{x}_v$. If we parametrize the curve by taking $v = t$, $0 \le t \le 2\pi$, then we have (see Example 1 of Chapter 2, Section 3)

$$\nabla_{\boldsymbol{\alpha}'}\mathbf{e}_1 = \nabla_{\boldsymbol{\alpha}'}\mathbf{x}_u = (\mathbf{x}_{uv})^{\parallel} = \cot u_0 \mathbf{x}_v = \cos u_0 \mathbf{e}_2,$$

and so $\phi_{12} = \cos u_0$. Therefore, the holonomy around the latitude circle (oriented counterclockwise) is $\Delta\psi = -\int_0^{2\pi} \cos u_0\, dt = -2\pi\cos u_0$, confirming our previous results.

Note that if we wish to parametrize the curve by arclength (as will be important shortly), we take $s = (\sin u_0)v$, $0 \le s \le 2\pi \sin u_0$. Then, with respect to this parametrization, we have $\phi_{12}(s) = \cot u_0$. (Why?)

For completeness, we can use Proposition 1.1 to calculate ϕ_{12} as well: With $E = 1$, $G = \sin^2 u$, $u = u_0$, and $v(s) = s/\sin u_0$, we have $\phi_{12} = \dfrac{1}{2\sin u_0}\left(2\sin u_0 \cos u_0 \cdot \dfrac{1}{\sin u_0}\right) = \cot u_0$, as before. ∇

Suppose now that $\boldsymbol{\alpha}$ is an arclength-parametrized curve and let's write $\boldsymbol{\alpha}(s) = \mathbf{x}(u(s), v(s))$ and $\mathbf{T}(s) = \boldsymbol{\alpha}'(s) = \cos\theta(s)\mathbf{e}_1 + \sin\theta(s)\mathbf{e}_2$, $s \in [0, L]$, for a $\mathcal{C}^1$ function $\theta(s)$ (cf. Lemma 3.6 of Chapter 1), as indicated in Figure 1.2. A formula fundamental for the rest of our work is the following:

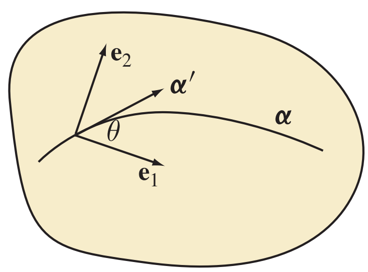

FIGURE 1.2

Proposition 1.3. *When $\boldsymbol{\alpha}$ is an arclength-parametrized curve, the geodesic curvature of $\boldsymbol{\alpha}$ is given by*

$$\kappa_g(s) = \phi_{12}(s) + \theta'(s) = \frac{1}{2\sqrt{EG}}\big(-E_v u'(s) + G_u v'(s)\big) + \theta'(s).$$

Proof. Recall that $\kappa_g = \kappa\mathbf{N}\cdot(\mathbf{n}\times\mathbf{T}) = \mathbf{T}'\cdot(\mathbf{n}\times\mathbf{T})$. Now, since $\mathbf{T} = \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2$, $\mathbf{n}\times\mathbf{T} = -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2$ (why?), and so

$$\begin{aligned}
\kappa_g &= \nabla_{\mathbf{T}}\mathbf{T}\cdot(-\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2)\\
&= \nabla_{\mathbf{T}}(\cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2)\cdot(-\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2)\\
&= \big(\cos\theta\nabla_{\mathbf{T}}\mathbf{e}_1 + \sin\theta\nabla_{\mathbf{T}}\mathbf{e}_2\big)\cdot(-\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2) + \big((-\sin\theta)\theta'(-\sin\theta) + (\cos\theta)\theta'(\cos\theta)\big)\\
&= (\cos^2\theta + \sin^2\theta)(\phi_{12} + \theta') = \phi_{12} + \theta',
\end{aligned}$$

as required. Now the result follows by applying Proposition 1.1 when $\boldsymbol{\alpha}$ is arclength-parametrized. $\square$

Remark. The first equality in Proposition 1.3 should not be surprising in the least. Curvature of a plane curve measures the rate at which its unit tangent vector turns relative to a fixed reference direction. Similarly, the geodesic curvature of a curve in a surface measures the rate at which its unit tangent vector turns relative to a parallel vector field along the curve; θ' measures its turning relative to $\mathbf{e}_1$, which is itself turning at a rate given by ϕ_{12}, so the geodesic curvature is the sum of those two rates.

Now suppose that $\boldsymbol{\alpha}$ is a *closed* curve bounding a region $R \subset M$. We denote the boundary of R by ∂R. Then by Green's Theorem (see Theorem 2.6 of the Appendix), we have

$$(\dagger)\qquad \begin{aligned}\int_0^L \phi_{12}(s)ds &= \int_0^L \frac{1}{2\sqrt{EG}}\big(-E_v u'(s) + G_u v'(s)\big)ds = \int_{\partial R} \frac{1}{2\sqrt{EG}}\big(-E_v du + G_u dv\big)\\ &= \iint_R \left(\Big(\frac{E_v}{2\sqrt{EG}}\Big)_v + \Big(\frac{G_u}{2\sqrt{EG}}\Big)_u\right) dudv\\ &= \iint_R \frac{1}{2\sqrt{EG}}\left(\Big(\frac{E_v}{\sqrt{EG}}\Big)_v + \Big(\frac{G_u}{\sqrt{EG}}\Big)_u\right) \underbrace{\sqrt{EG}dudv}_{dA}\\ &= -\iint_R KdA\end{aligned}$$

by the formula $(*)$ for Gaussian curvature on p. 60. (Recall from the end of Section 1 of Chapter 2 that the element of surface area on a parametrized surface is given by $dA = \|\mathbf{x}_u \times \mathbf{x}_v\| dudv = \sqrt{EG - F^2}dudv$.)

We now see that Gaussian curvature and holonomy are intimately related:

Corollary 1.4. *When R is a region with smooth boundary and lying in an orthogonal parametrization, the holonomy around ∂R is $\Delta\psi = \iint_R KdA$.*

Proof. This follows immediately from Proposition 1.2 and the formula $(\dagger)$ above. $\square$

We conclude further from Proposition 1.3 that

$$\int_{\partial R} \kappa_g ds = \int_{\partial R} \phi_{12} ds + \underbrace{\theta(L) - \theta(0)}_{\Delta\theta},$$

so the total angle through which the tangent vector to ∂R turns is given by

$$\Delta\theta = \int_{\partial R} \kappa_g ds + \iint_R KdA.$$

Now, when R is simply connected (i.e., can be continuously deformed to a point), it is not too surprising that $\Delta\theta = 2\pi$. Intuitively, as we shrink the curve to a point, $\mathbf{e}_1$ becomes almost constant along the curve, but the tangent vector must make one full rotation (as a consequence of the Hopf Umlaufsatz, Theorem 3.5 of Chapter 1). Since $\Delta\theta$ is an integral multiple of 2π that varies continuously as we deform the curve, it must stay equal to 2π throughout.

Corollary 1.5. *If R is a simply connected region lying in an orthogonal parametrization and whose boundary curve is a geodesic, then $\iint_R KdA = \Delta\theta = 2\pi$.*

Example 2. We take R to be the upper hemisphere and use the usual spherical coordinates parametrization. Then the unit tangent vector along ∂R is $\mathbf{e}_2$ everywhere, so $\Delta\theta = 0$, in contradiction with Corollary