Remark. The first equality in Proposition 1.3 should not be surprising in the least. Curvature of a plane curve measures the rate at which its unit tangent vector turns relative to a fixed reference direction. Similarly, the geodesic curvature of a curve in a surface measures the rate at which its unit tangent vector turns relative to a parallel vector field along the curve; θ' measures its turning relative to $\mathbf{e}_1$, which is itself turning at a rate given by ϕ_{12}, so the geodesic curvature is the sum of those two rates.

Now suppose that $\boldsymbol{\alpha}$ is a *closed* curve bounding a region $R \subset M$. We denote the boundary of R by ∂R. Then by Green's Theorem (see Theorem 2.6 of the Appendix), we have

$$
\begin{aligned}
\int_0^L \phi_{12}(s)ds &= \int_0^L \frac{1}{2\sqrt{EG}}\big(-E_v u'(s) + G_u v'(s)\big)ds = \int_{\partial R} \frac{1}{2\sqrt{EG}}\big(-E_v du + G_u dv\big)\\
&= \iint_R \left(\Big(\frac{E_v}{2\sqrt{EG}}\Big)_v + \Big(\frac{G_u}{2\sqrt{EG}}\Big)_u\right) du dv\\
&= \iint_R \frac{1}{2\sqrt{EG}}\left(\Big(\frac{E_v}{\sqrt{EG}}\Big)_v + \Big(\frac{G_u}{\sqrt{EG}}\Big)_u\right) \underbrace{\sqrt{EG}du dv}_{dA}\\
&= -\iint_R K dA
\end{aligned} \tag{†}
$$

by the formula (∗) for Gaussian curvature on p. 60. (Recall from the end of Section 1 of Chapter 2 that the element of surface area on a parametrized surface is given by $dA = \|\mathbf{x}_u \times \mathbf{x}_v\| du dv = \sqrt{EG - F^2} du dv$.)

We now see that Gaussian curvature and holonomy are intimately related:

Corollary 1.4. *When R is a region with smooth boundary and lying in an orthogonal parametrization, the holonomy around ∂R is $\Delta\psi = \iint_R K dA$.*

Proof. This follows immediately from Proposition 1.2 and the formula (†) above. □

We conclude further from Proposition 1.3 that

$$\int_{\partial R} \kappa_g ds = \int_{\partial R} \phi_{12} ds + \underbrace{\theta(L) - \theta(0)}_{\Delta\theta},$$

so the total angle through which the tangent vector to ∂R turns is given by

$$\Delta\theta = \int_{\partial R} \kappa_g ds + \iint_R K dA.$$

Now, when R is simply connected (i.e., can be continuously deformed to a point), it is not too surprising that $\Delta\theta = 2\pi$. Intuitively, as we shrink the curve to a point, $\mathbf{e}_1$ becomes almost constant along the curve, but the tangent vector must make one full rotation (as a consequence of the Hopf Umlaufsatz, Theorem 3.5 of Chapter 1). Since $\Delta\theta$ is an integral multiple of 2π that varies continuously as we deform the curve, it must stay equal to 2π throughout.

Corollary 1.5. *If R is a simply connected region lying in an orthogonal parametrization and whose boundary curve is a geodesic, then $\iint_R K dA = \Delta\theta = 2\pi$.*

Example 2. We take R to be the upper hemisphere and use the usual spherical coordinates parametrization. Then the unit tangent vector along ∂R is $\mathbf{e}_2$ everywhere, so $\Delta\theta = 0$, in contradiction with Corollary

1.5. Alternatively, $C = \partial R$ is a geodesic, so there should be zero holonomy around C (computed with respect to this framing).

How do we resolve this paradox? Well, although we've been sloppy about this point, the spherical coordinates parametrization actually fails at the north pole (since $\mathbf{x}_v = \mathbf{0}$). Indeed, there is no framing of the upper hemisphere with $\mathbf{e}_2$ everywhere tangent to the equator. However, the reader can rest assured that there *is* some orthogonal parametrization of the upper hemisphere, e.g., by stereographic projection from the south pole (cf. Example 1(e) in Section 1 of Chapter 2). ∇

Remark. In more advanced courses, the holonomy around the closed curve $\boldsymbol{\alpha}$ is interpreted as a rotation of the tangent plane of M at $\boldsymbol{\alpha}(0)$. That is, what matters is $\Delta\psi \pmod{2\pi}$, i.e., the change in angle disregarding multiples of 2π. This quantity does not depend on the choice of framing $\mathbf{e}_1, \mathbf{e}_2$.

We now set to work on one of the crowning results of surface theory.

Theorem 1.6 (Local Gauss-Bonnet)**.** *Suppose R is a simply connected region with piecewise smooth boundary and lying in an orthogonal parametrization. If $C = \partial R$ has exterior angles ϵ_j, $j = 1, \dots, \ell$, then*

$$\int_{\partial R} \kappa_g \, ds + \iint_R K \, dA + \sum_{j=1}^{\ell} \epsilon_j = 2\pi .$$

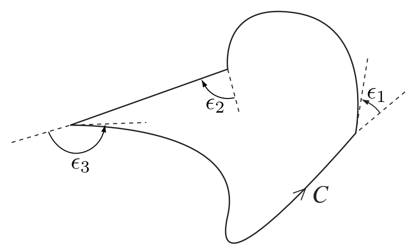


FIGURE 1.3

Note, as we indicate in Figure 1.3, that we measure exterior angles so that $|\epsilon_j| \le \pi$ for all j.

Proof. If ∂R is smooth, then from our earlier discussion we infer that

$$\int_{\partial R} \kappa_g \, ds + \iint_R K \, dA = \Delta\theta = 2\pi .$$

But when ∂R has corners, the unit tangent vector turns *less* by the amount $\sum_{j=1}^{\ell} \epsilon_j$, so the result follows. (Technically, what we need is the correction of the Hopf Umlaufsatz when the curve has corners. See Exercise 1.3.12.) $\square$

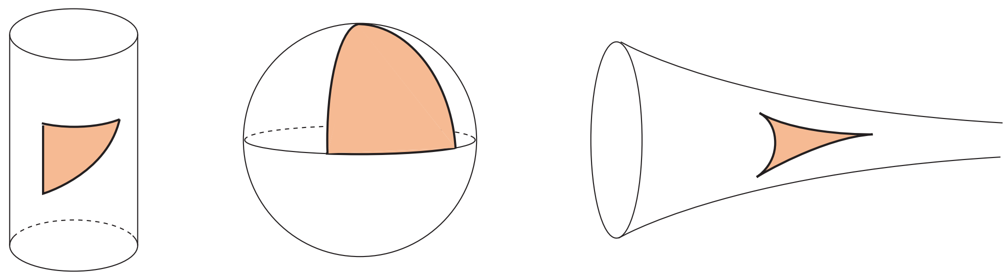
Corollary 1.7. *For a geodesic triangle (i.e., a region whose boundary consists of three geodesic segments) R with* interior *angles $\iota_1, \iota_2, \iota_3$, we have $\iint_R K \, dA = (\iota_1 + \iota_2 + \iota_3) - \pi$, the* angle excess.

Proof. Since the boundary consists of geodesic segments, the geodesic curvature integral drops out, and we are left with

$$\iint_R K \, dA = 2\pi - \sum_{j=1}^{3} \epsilon_j = 2\pi - \sum_{j=1}^{3} (\pi - \iota_j) = \sum_{j=1}^{3} \iota_j - \pi ,$$

as required. $\square$

Remark. It is worthwhile to consider the three special cases $K = 0$, $K = 1$, $K = -1$, as pictured in Figure 1.4. When M is flat, the sum of the angles of a triangle is π, as in the Euclidean case. When M



$K = 0$ $K = 1$ $K = -1$

FIGURE 1.4

is positively curved, it takes more than π for the triangle to close up, and when M is negatively curved, it takes less. Intuitively, this is because geodesics seem to "bow out" when $K > 0$ and "bow in" when $K < 0$ (cf. Exercise 3.2.17).

Example 3. Let's consider Theorem 1.6 in the case of a spherical cap, as shown in Figure 1.5. Using the usual spherical coordinates parametrization, we have $0 \le u \le u_0$. By Proposition 1.3 and Example 1,

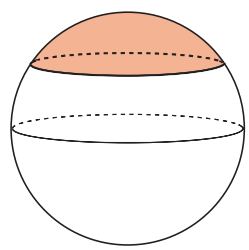


FIGURE 1.5

since $\theta = \pi/2$ along the v-curve, we have $\kappa_g = \phi_{12}(s) = \cot u_0$ (cf. also Exercise 2.4.4). Therefore, we have

$$\iint_R K\,dA = 2\pi - \int_{\partial R} \kappa_g\,ds = 2\pi(1 - \cos u_0),$$

which checks, of course, since $K = 1$ and the area of this cap is indeed

$$\int_0^{2\pi}\int_0^{u_0} \sin u\,du\,dv = 2\pi(1 - \cos u_0). \qquad \nabla$$

Remark. Notice that the sign of κ_g depends on both the orientation of $\boldsymbol{\alpha}$ and the orientation of the surface. If we rescale the surface by a factor of c, then the integral $\int_{\partial R} \kappa_g\,ds$ does not change, as the arclength changes by a factor of c and the geodesic curvature by a factor of $1/c$. Similarly, the integral $\iint_R K\,dA$ does not change when we rescale the surface: Area changes by a factor of c^2 and Gaussian curvature changes by a factor of $1/c^2$.

We now come to one of the crowning results of modern-day mathematics, one which has led to much subsequent research and generalization. We say a surface $M \subset \mathbb{R}^3$ is *oriented* if we have chosen a continuous unit normal field defined everywhere on M. We now consider a compact, oriented surface with

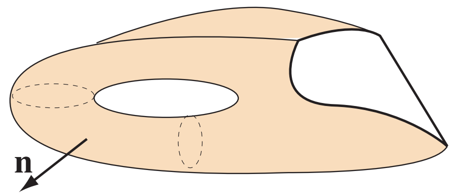


FIGURE 1.6

piecewise-smooth boundary, as pictured in Figure 1.6. T. Radó proved in 1925 that any such surface M can be triangulated. That is, we may write $M = \bigcup_{\lambda=1}^{m} \Delta_\lambda$ where

(i) Δ_λ is the image of a triangle under an (orientation-preserving) orthogonal parametrization;
(ii) $\Delta_\lambda \cap \Delta_\mu$ $(\lambda \neq \mu)$ is either empty, a single vertex, or a single edge;
(iii) when $\Delta_\lambda \cap \Delta_\mu$ consists of a single edge, the orientations of the edge are opposite in Δ_λ and Δ_μ; and
(iv) at most one edge of Δ_λ is contained in the boundary of M.

We now make a standard

Definition. Given a triangulation $\mathcal{T}$ of a surface M with V vertices, E edges, and F faces, we define the *Euler characteristic* $\chi(M, \mathcal{T}) = V - E + F$.

Example 4. We can triangulate a disk as shown in Figure 1.7, obtaining $\chi = 1$. Without being so

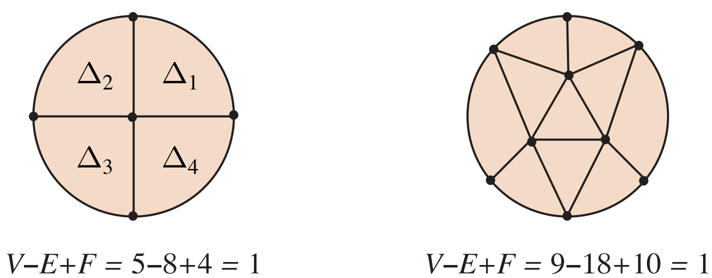


FIGURE 1.7

pedantic as to require that each Δ_λ be the image of a triangle under an orthogonal parametrization, we might just think of the disk as a single triangle with its edges puffed out; then we would have $\chi = V - E + F = 3 - 3 + 1 = 1$, as well. We leave it to the reader to triangulate a sphere and check that $\chi(\Sigma, \mathcal{T}) = 2$. ∇

Remark. It's important to note that by choosing the orientations on the "triangles" Δ_λ compatibly, we get an orientation on the boundary of M. That is, a choice of $\mathbf{n}$ on M determines which direction we proceed on ∂M. This is precisely the case any time one deals with Green's Theorem (or its generalization to oriented surfaces, Stokes's Theorem). Nevertheless, following up on the Remark on p. 85, the sign of κ_g on ∂M is independent of the choice of orientation on M, for, if we change $\mathbf{n}$ to $-\mathbf{n}$, the orientation on ∂M switches and $\mathbf{n} \times \mathbf{T}$ stays the same.

The beautiful result to which we've been headed is now the following

Theorem 1.8 (Global Gauss-Bonnet). *Let M be a compact, oriented surface with piecewise-smooth boundary, equipped with a triangulation $\mathcal{T}$ as above. If ϵ_k, $k = 1, \dots, \ell$, are the exterior angles of ∂M, then*

$$\int_{\partial M} \kappa_g \, ds + \iint_M K \, dA + \sum_{k=1}^{\ell} \epsilon_k = 2\pi \chi(M, \mathcal{T}).$$

Proof. As we illustrate in Figure 1.8, we will distinguish vertices on the boundary and in the interior, denoting the respective total numbers by V_b and V_i. Similarly, we distinguish among edges on the boundary, edges in the interior, and edges that join a boundary vertex to an interior vertex; we denote the respective

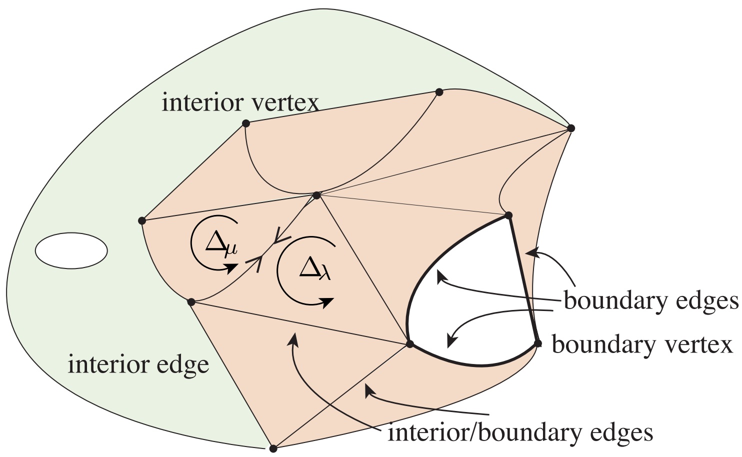


FIGURE 1.8

numbers of these by E_b, E_i, and E_{ib}. Now observe that

$$\iint_M K \, dA = \sum_{\lambda=1}^{m} \iint_{\Delta_\lambda} K \, dA$$

since all the orientations are compatible, and

$$\int_{\partial R} \kappa_g \, ds = \sum_{\lambda=1}^{m} \int_{\partial \Delta_\lambda} \kappa_g \, ds$$

because the line integrals over interior and interior/boundary edges cancel in pairs (recall that κ_g changes sign when we reverse the orientation of the curve). Let $\epsilon_{\lambda j}$, $j = 1, 2, 3$, denote the exterior angles of the "triangle" Δ_λ. Then, applying Theorem 1.6 to Δ_λ, we have

$$\int_{\partial \Delta_\lambda} \kappa_g \, ds + \iint_{\Delta_\lambda} K \, dA + \sum_{j=1}^{3} \epsilon_{\lambda j} = 2\pi,$$

and now, summing over the triangles, we obtain

$$\int_{\partial M} \kappa_g \, ds + \iint_M K \, dA + \sum_{\lambda=1}^{m} \sum_{j=1}^{3} \epsilon_{\lambda j} = 2\pi m = 2\pi F.$$

Now we must do some careful accounting: Letting $\iota_{\lambda j}$ denote the respective interior angles of triangle Δ_λ, we have

$$(*) \qquad \sum_{\substack{\text{interior}\\ \text{vertices}}} \epsilon_{\lambda j} = \sum_{\substack{\text{interior}\\ \text{vertices}}} (\pi - \iota_{\lambda j}) = \pi(2E_i + E_{ib}) - 2\pi V_i$$

inasmuch as each interior edge contributes two interior vertices, whereas each interior/boundary edge contributes just one, and the interior angles at each interior vertex sum to 2π. Next,

$$(**) \qquad \sum_{\substack{\text{boundary}\\ \text{vertices}}} \epsilon_{\lambda j} = \pi E_{ib} + \sum_{k=1}^{\ell} \epsilon_k .$$

To see this, we reason as follows. Given a boundary vertex v, denote by a superscript (v) the relevant angle or number for which the vertex v is involved. Note first of all that any boundary vertex v is contained in $E_{ib}^{(v)} + 1$ faces. Moreover, for a fixed boundary vertex v,

$$\sum \iota_{\lambda j}^{(v)} = \begin{cases} \pi, & v \text{ a smooth boundary vertex} \\ \pi - \epsilon_k, & v \text{ a corner of } \partial M \text{ with exterior angle } \epsilon_k \end{cases} .$$

Thus,

$$\begin{aligned} \sum_{\substack{\text{boundary}\\ \text{vertices}}} \epsilon_{\lambda j} &= \sum_{\substack{\text{boundary}\\ \text{vertices } v}} (\pi - \iota_{\lambda j}) = \sum_{\substack{\text{boundary}\\ \text{vertices } v}} \pi(E_{ib}^{(v)} + 1) - \Big(\sum_{v \text{ smooth}} \iota_{\lambda j} + \sum_{v \text{ corner}} \iota_{\lambda j} \Big) \\ &= \pi E_{ib} + \sum_{k=1}^{\ell} \epsilon_k . \end{aligned}$$

Adding equations $(*)$ and $(**)$ yields

$$\sum_{\lambda, j} \epsilon_{\lambda j} = \sum_{\substack{\text{interior}\\ \text{vertices}}} \epsilon_{\lambda j} + \sum_{\substack{\text{boundary}\\ \text{vertices}}} \epsilon_{\lambda j} = 2\pi(E_i + E_{ib} - V_i) + \sum_{k=1}^{\ell} \epsilon_k .$$

At long last, therefore, our reckoning concludes:

$$\begin{aligned} \int_{\partial M} \kappa_g \, ds + \iint_M K \, dA + \sum_{k=1}^{\ell} \epsilon_k &= 2\pi\big(F - (E_i + E_{ib}) + V_i\big) \\ &= 2\pi\big(F - (E_i + E_{ib} + E_b) + (V_i + V_b)\big) = 2\pi(V - E + F) \\ &= 2\pi\chi(M, \mathcal{T}). \end{aligned}$$

(Note that because the boundary curve ∂M is closed, we have $V_b = E_b$.) $\square$

We now derive some interesting conclusions:

Corollary 1.9. *The Euler characteristic $\chi(M, \mathcal{T})$ does not depend on the triangulation $\mathcal{T}$ of M.*

Proof. The left-hand side of the equality in Theorem 1.8 has nothing whatsoever to do with the triangulation. $\square$

It is therefore legitimate to denote the Euler characteristic by $\chi(M)$, with no reference to the triangulation. It is proved in a course in algebraic topology that the Euler characteristic is a "topological invariant"; i.e., if we deform the surface M in a bijective, continuous manner (so as to obtain a *homeomorphic* surface), the Euler characteristic does not change. We therefore deduce:

Corollary 1.10. *The quantity*

$$\int_{\partial M} \kappa_g \, ds + \iint_M K \, dA + \sum_{k=1}^{\ell} \epsilon_k$$

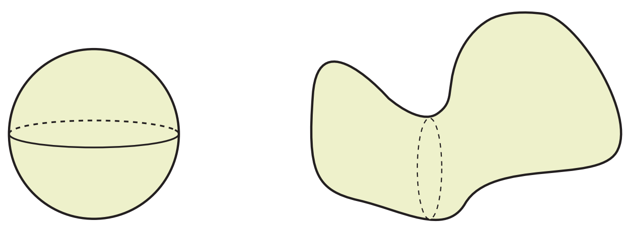
is a topological invariant, i.e., does not change as we deform the surface M.

In particular, in the event that $\partial M = \emptyset$ (so many people refer to the surface M as a *closed* surface), we have

Corollary 1.11. *When M is a compact, oriented surface without boundary, we have*

$$\iint_M K \, dA = 2\pi\chi(M).$$

It is very interesting that the *total curvature* does not change as we deform the surface, for example, as shown in Figure 1.9. In a topology course, one proves that any compact, oriented surface without boundary must



$\iint_M K \, dA = 4\pi$

FIGURE 1.9

have the topological type of a sphere or of a g-holed torus for some positive integer g. Thus (cf. Exercise 4), the possible Euler characteristics of such a surface are $2, 0, -2, -4, \ldots$; moreover, the integral $\iint_M K \, dA$ determines the topological type of the surface.

We conclude this section with a few applications of the Gauss-Bonnet Theorem.

Example 5. Suppose M is a surface of nonpositive Gaussian curvature. Then there cannot be a geodesic 2-gon R on M that bounds a simply connected region. For if there were, by Theorem 1.6 we would have

$$0 \geq \iint_R K \, dA = 2\pi - (\epsilon_1 + \epsilon_2) > 0,$$

which is a contradiction. (Note that the exterior angles must be strictly less than π because there is a unique (smooth) geodesic with a given tangent direction.) $\triangledown$

Example 6. Suppose M is topologically equivalent to a cylinder and its Gaussian curvature is negative. Then there is at most one simple closed geodesic in M. Note, first, as indicated in Figure 1.10, that if there is a simple closed geodesic $\boldsymbol{\alpha}$, either it must separate M into two unbounded pieces or else it bounds

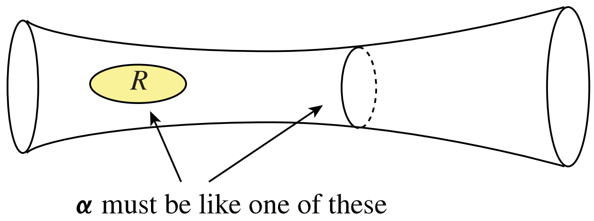


FIGURE 1.10

a disk R, in which case we would have $0 > \iint_R K\,dA = 2\pi\chi(R) = 2\pi$, which is a contradiction. On the other hand, suppose there were two. If they don't intersect, then they bound a cylinder R and we get $0 > \iint_R K\,dA = 2\pi\chi(R) = 0$, which is a contradiction. If they do intersect, then we we have a geodesic 2-gon bounding a simply connected region, which cannot happen by Example 5. $\triangledown$

EXERCISES 3.1

1. Compute the holonomy around the parallel $u = u_0$ (and indicate which direction the rotation occurs from the viewpoint of an observer away from the surface down the x-axis) on
   *a. the torus $\mathbf{x}(u, v) = \big((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u\big)$
   b. the paraboloid $\mathbf{x}(u, v) = (u\cos v, u\sin v, u^2)$
   c. the catenoid $\mathbf{x}(u, v) = (\cosh u\cos v, \cosh u\sin v, u)$

*2. Determine whether there can be a (smooth) closed geodesic on a surface when
   a. $K > 0$
   b. $K = 0$
   c. $K < 0$

   If the closed geodesic can bound a simply connected region, give an example.

3. Calculate the Gaussian curvature of a torus (as parametrized in Example 1(c) of Section 1 of Chapter 2) and verify Corollary 1.11.

4. a. Triangulate a cylinder, a sphere, a torus, and a two-holed torus; verify that $\chi = 0, 2, 0$, and -2, respectively. Pay particular attention to condition (ii) in the definition of triangulation.
   b. Prove by induction that a g-holed torus has $\chi = 2 - 2g$.

5. Suppose M is a compact, oriented surface without boundary that is *not* of the topological type of a sphere. Prove that there are points in M where Gaussian curvature is positive, zero, and negative.

6. Consider a surface with $K > 0$ that is topologically a cylinder. Prove that there cannot be two disjoint simple closed geodesics both going around the neck of the surface.

7. Suppose M and M^* are locally isometric and compatibly oriented. Use Proposition 1.3 to prove that if $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}^*$ are corresponding arclength-parametrized curves, then their geodesic curvatures are equal at corresponding points.