Test vectors, path and cycle graphs.

An easy way to get upper bounds on $\lambda_2(G)$ is to plug vectors into the Rayleigh quotient.

Proposition. If $\vec{\alpha} \in \mathbb{R}^{v}$, and $\langle \vec{\alpha}, \vec{1} \rangle = 0$

$$\lambda_2(G) \leq \frac{\langle \vec{\alpha}, \vec{\alpha} \rangle_{L_G}}{\langle \vec{\alpha}, \vec{\alpha} \rangle} = \frac{\sum_{a \sim b} (\vec{\alpha}(a) - \vec{\alpha}(b))^2}{\sum_a \vec{\alpha}(a)^2}$$

Proof. Since $L_G \vec{1} = \vec{0}$, $\lambda_2 = \min_{\langle \vec{\beta}, \vec{1} \rangle = 0} \frac{\langle \vec{\beta}, \vec{\beta} \rangle_{L_G}}{\langle \vec{\beta}, \vec{\beta} \rangle}$ by Courant-Fischer. $\square$

Example. Let P_v be the path graph on v vertices $1 - 2 - \cdots - v$. Let

$$\vec{\alpha}(a) = (v+1) - 2a.$$

We see that

$$\langle \vec{\alpha}, \vec{1} \rangle = \sum_{a=1}^{v} (v+1) - 2a$$

$$= v(v+1) - 2\sum_{a=1}^{v} a$$

$$= v(v+1) - \cancel{2}\,\frac{v(v+1)}{\cancel{2}} = 0.$$

Thus we know

$$\lambda_2(P_v) \leq \frac{\sum_{a=1}^{v-1} (\vec{\alpha}(a) - \vec{\alpha}(a+1))^2}{\sum_{a=1}^{v} \vec{\alpha}(a)^2}$$

$$\leq \frac{\sum_{a=1}^{v-1} 2^2}{\sum_{a=1}^{v} ((v+1) - 2a)^2}$$

$$\cancel{\leq \frac{4(v-1)}{\sum_{a=v+}^{1} ((v+1) - 2a)^2}}$$

$$\cancel{\leq \frac{4(v-1)}{\sum_{i=1} (2i+1 - \ldots}}$$

We can rewrite the denominator

$$\sum_{a=1}^{v}(2a-(v+1))^2 = 4\sum_{a=1}^{v}a^2 - 4(v+1)\sum_{a=1}^{v}a + \sum_{a=1}^{v}(v+1)^2$$

$$= 4\,\frac{v(v+1)(2v+1)}{6} - \frac{4(v+1)v(v+1)}{2} + v(v+1)^2$$

$$= \frac{v(v+1)}{6}\left(4(2v+1) - 12(v+1) + 6(v+1)\right)$$

$$= \frac{v(v+1)}{6}\left(4(2v+1) - 6(v+1)\right)$$

$$= \frac{v(v+1)}{6}\left(8v+4-6v-6\right)$$

$$= \frac{v(v+1)}{6}(2v-2)$$

$$= \frac{(v-1)v(v+1)}{3}$$

So we have

$$\lambda_2(P_v) \leq \frac{4\cancel{(v-1)}}{\frac{\cancel{(v-1)}v(v+1)}{3}} = \frac{12}{v(v+1)}$$

We can compare this bound to

$$\lambda_2(P_v) \leq \frac{\theta(S)}{1 - \frac{|S|}{v}} = \frac{v}{v-|S|} \cdot \frac{|\partial S|}{|S|}$$

where S is any subset of the vertices of P_v. It turns out that the right hand side is minimized when $S = \{1, \ldots, v/2\}$, and the value is

$$\lambda_2(P_v) \leq \frac{v}{v - v/2} \frac{1}{v/2} = \frac{v}{v^2/4} = \frac{4}{v}$$

This is the wrong order of magnitude!

We now compute the spectrum of the cycle graph and path graph exactly!

Proposition. The cycle or ring graph $C_v = 1 - 2 - \cdots - v - 1$ has eigenvectors

$$\vec{x}_k(a) = \cos\left(2\pi k \frac{a}{v}\right)$$

$$\vec{y}_k(a) = \sin\left(2\pi k \frac{a}{v}\right)$$

for $0 \leq k \leq v/2$ (except for $\vec{y}_0 = \vec{0}$ and, for v even, $\vec{y}_{v/2} = \vec{0}$). Eigenvectors $\vec{x}_k$ and $\vec{y}_k$ have eigenvalue

$$2 - 2\cos\left(2\pi \frac{k}{v}\right).$$

Proof. We know that

$$(L_{C_v} \vec{X}_k)(a) = \sum_{b \sim a} \vec{X}_k(a) - \vec{X}_k(b)$$

$$= 2\vec{X}_k(a) - \vec{X}_k(a+1) - \vec{X}_k(a-1)$$

$$= 2\cos\left(2\pi k \frac{a}{n_v}\right) - 2\cos\left(2\pi k \frac{a}{n_v} + 2\pi k \frac{1}{n_v}\right) - 2\cos\left(2\pi k \frac{a}{n_v} - 2\pi k \frac{1}{n_v}\right)$$

If we let $\theta = 2\pi k \frac{a}{n_v}$ and $\varphi = 2\pi k \frac{1}{n_v}$, this is

$$2\cos(\theta) - \cos(\theta + \varphi) - \cos(\theta - \varphi)$$

Recalling that $\cos(a+b) = \cos a \cos b - \sin a \sin b$, we simplify this as

$$2\cos\theta - \cos\theta\cos\varphi + \cancel{\sin\theta\sin\varphi} - \cos\theta\cos(-\varphi) + \cancel{\sin\theta\sin(-\varphi)}$$

$$= 2\cos\theta(1-\cos\varphi)$$

so

$$L_{C_v}\vec{x}_k = (2-2\cos\varphi)\,\vec{x}_k$$

$$= (2-2\cos(2\pi k/v))$$

The argument for $\vec{y}_k$ is similar. $\square$

Thus

$$\lambda_2(C_v) = 2-2\cos\left(\frac{2\pi}{v}\right)$$

$$\approx 2-2\left(1-\left(\frac{2\pi}{v}\right)^2 \cdot\right)$$

$$\approx \frac{4\pi^2}{v^2}$$

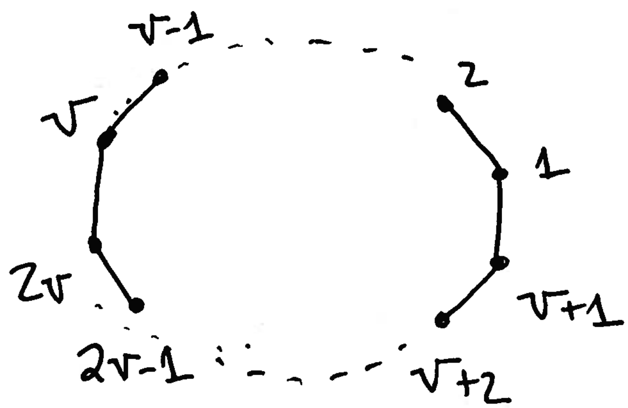
We're now going to use this to compute the eigenvalues and eigenvectors of the path graph.

Proposition. If P_v is the path graph $1 - 2 - \cdots - v$, then L_{P_v} has eigenvalues

$\lambda_k = 2 - 2\cos(\pi k/v)$ and eigenvectors

$\vec{x}_k = \cos(\pi k a/v - \pi k/2v)$, for $k \in 0, \ldots, v-1$.

Proof. We start by renumbering the vertices of the cycle graph C_{2v}



You'll prove for homework that with this numbering

$$(I_v \; I_v) L_{C_{2v}} \begin{pmatrix} I_v \\ I_v \end{pmatrix} = 2 L_{P_v}$$

Now suppose we have an eigenvector $\vec{\alpha}$ of C_{2v} so that $\vec{\alpha}$ has eigenvalue λ,

$$\vec{\alpha}(a) = \vec{\alpha}(a+v)$$

for $a \in 1, \ldots, v$. We define $\vec{\varphi}(a) = \vec{\alpha}(a)$ for $a \in 1, \ldots, v$. Then

$$\begin{pmatrix} I_v \\ I_v \end{pmatrix} \vec{\varphi} = \vec{\alpha}, \quad \text{so} \quad L_{C_{2v}} \begin{pmatrix} I_v \\ I_v \end{pmatrix} \vec{\varphi} = \lambda \begin{pmatrix} I_v \\ I_v \end{pmatrix} \vec{\varphi}$$

and

$$(I_v \; I_v) L_{C_{2v}} \begin{pmatrix} I_v \\ I_v \end{pmatrix} \vec{\varphi} = \lambda (I_v I_v) \begin{pmatrix} I_v \\ I_v \end{pmatrix} \vec{\varphi}$$

$$= 2\lambda\vec{\varphi}$$

Thus

$$2L_{P_v}\vec{\varphi} = 2\lambda\vec{\varphi}$$

and $\vec{\varphi}$ is an eigenvector of L_{P_v} of eigenvalue λ.

We'll check for homework that there is one such $\vec{\alpha}$ in each eigenspace of C_{2v}, ~~so~~ ~~the~~ ~~e~~ (as given in the statement of the theorem), so the eigenvalues of P_v are those of C_{2v}:

$$2 - 2\cos(\cancel{2}\pi k/\cancel{2}v) = 2 - 2\cos(\pi k/v). \quad \square$$